Kalman Filter

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Probability

- The probability
  - That the outcome of a discrete event (e.g., a coin flip) will favor a particular event

$$p(A) = \frac{possible\ outcome\ favoring\ event\ A}{Total\ number\ of\ possible\ outcomes}$$

  - Of an outcome favoring either A or B

$$p(A \cup B) = p(A) + p(B)$$

    - If the probability of two outcomes is independent

$$p(A \cap B) = p(A)p(B)$$

  - Of outcome A given an occurrence of outcome B

$$p(A|B) = \frac{p(A \cap B)}{p(B)}, conditionally\ probability$$

Random Variables -1

- A random variable
  - Usually written X, is a variable whose possible values are numerical outcomes of a random phenomenon
  - Two types: Discrete and continuous
- Discrete random variables
  - One which may take on either a finite or at most a countably infinite set of discrete values
  - Ex: The number of children in a family, the attendance at a cinema

Random Variables -2

- Continuous random variables
  - One which takes on values that vary continuously within one or more real intervals, and have a cumulative distribution function (CDF) that is absolutely continuous
  - Having an uncountable infinite number of possible values, all of which have probability 0, though ranges of such values can have nonzero probability
  - Are usually measurements
  - Ex: Height, weight, etc

Random Variables -3

- The probability distribution
  - Or probability function, or probability mass function
- The probability distribution of a discrete random variable
  - A list of probabilities associated with each of its possible values
  - Ex: A random variable X may take k values, with the probability that $X = X_i$ defined to be $P(X = X_i) = p_i$, the probabilities p_i must satisfy the following

$$0 \leq p_i \leq 1 \; for \; each \; i$$
$$p_1 + p_2 + \cdots + p_k = 1$$

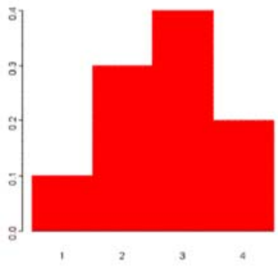
Random Variables -4

- A cumulative distribution function
  - Giving the probability that the random variable X is less than or equal to x, for every value x
  - Found by summing up the probabilities (for a discrete random variable)

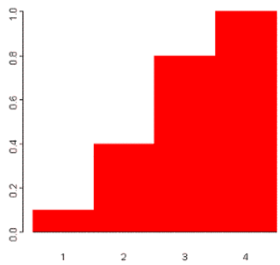
Random Variables -5

- Ex: X

Outcome	1	2	3	4
Probability	0.1	0.3	0.4	0.2



Probability distribution



Cumulative distribution

  - The probability of X equal to 2 or 3

$$p(X = 2 \cup X = 3) = p(X = 2) + p(X = 3) = 0.3 + 0.4 = 0.7$$

  - The probability that X is greater than 1

$$p(X > 1) = 1 - p(X = 1) = 1 - 0.1 = 0.9$$

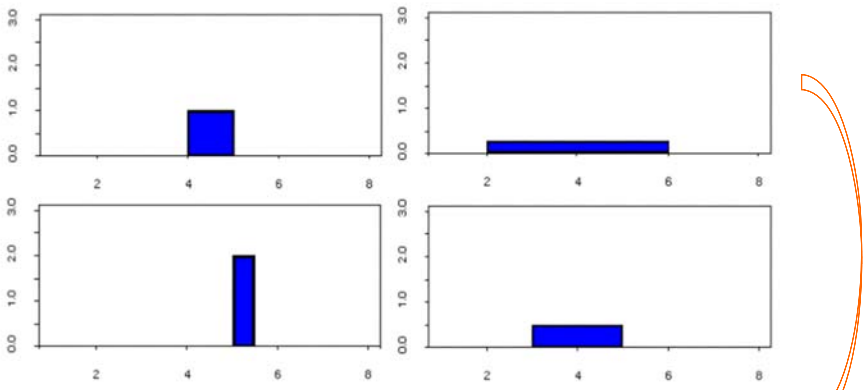
Random Variables -6

- The probability distribution of a continuous random variable
  - Not defined at specific values; defined over an interval of values, and is represented by an integral
  - Ex: A random variable X may take all values over an interval of real numbers
    - The probability that X is in the set of outcomes A, $p(A)$, is defined to be the area above A and under a curve (i.e., known as a density curve), which must satisfy the following

$$\forall x, p(x) \geq 0$$
$$\int p(x)dx = 1$$

Random Variables -7

- Ex: Uniform distribution
  - An interval of numbers (a,b) has a continuous distribution
  - Having an equal probability of being observed
  - Ex: intervals, (4,5), (2,6), (5,5.5), (3,5)



$$p(X \leq 3 \ \cup \ X \geq 5) = p(X \leq 3) + p(X \geq 5)$$
$$= (3-2) * 0.25 + (6-5) * 0.25 = 0.5$$

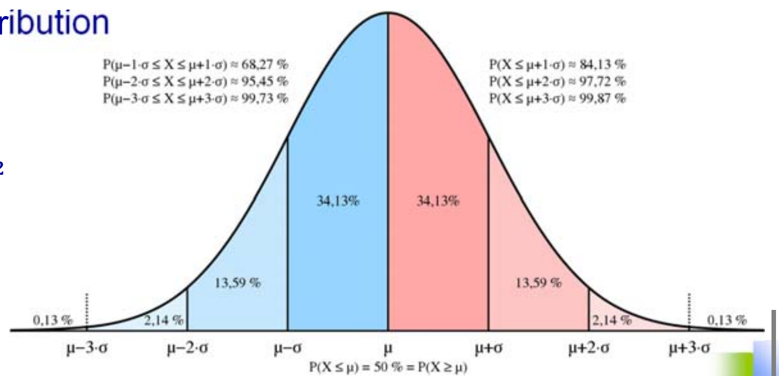
Random Variables -8

- Ex: Normal (or Gaussian or Gauss or Laplace-Gauss) distribution
  - Having a bell-shaped density curve described by its mean μ and standard deviation σ
  - Density curve: Symmetrical, centered about its mean, with its spread determined by its standard deviation

$$X \sim N(\mu, \sigma^2) \qquad p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

  - Standard normal distribution

$$X \sim N(0,1)$$
$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$



Random Variables -8

- Formulation

$$mean\ X = \mu = E[X]$$

$$\begin{aligned} variance\ X &= \sigma^2 = Var(X) = E[(X-\mu)^2] \\ &= E[(X-E[X])^2] = E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - 2E[X]E[X] + E[X]^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

Random Walk -1

- Description
  - A path that consists of a succession of random steps on some mathematical space such as the integers
  - Most often referred to a special category of Markov chains or Markov processes
    - A process for which predictions can be made regarding future outcomes based solely on its present state and--most importantly--such predictions are just as good as the ones that could be made knowing the process's full history
  - It is common to model the dynamics of the state as a Markov random walk in the highest estimated time derivatives, with zero-mean normally-distributed increments

Random Walk -2

- Formulation

$X[k] = X[k-1] + e[k] \qquad e[k] \sim N(0, \sigma^2)$

Previous state

$X[k] = (X[k-2] + e[k]) + e[k] = \cdots = X[0] + \sum_{j=1}^{k} e[k]$

$$
\begin{aligned}
mean = E[X[k]] &= E[X[0] + \textstyle\sum_{j=1}^{k} e[k]] \\
&= E[X[0]] + E[\textstyle\sum_{j=1}^{k} e[k]] \\
&= X[0] + \textstyle\sum_{j=1}^{k} E[e[k]] \\
&= X[0] + 0 = X[0]
\end{aligned}
$$

$$
\begin{aligned}
variance = Var(X) = E\left[\left(X[k] - E[X[k]]\right)^2\right] &= E[X[k]^2] - E[X[k]]^2 \\
&= E[(X[0] + \textstyle\sum e[k])(X[0] + \sum e[k])] - E[X[0]]^2 \\
&= E[X[0]^2 + 2X[0]\textstyle\sum e[k] + \sum e[k] \sum e[k]] - E[X[0]]^2 \\
&= E[X[0]^2] + 2X[0]\textstyle\sum \underline{E[e[k]]} + \underline{\sum\sum E[e[k]e[k]]} - E[X[0]]^2 \\
&= k\sigma^2
\end{aligned}
$$

$= 0$

$= \begin{cases} \sigma^2 & i = j \\ 0 & i \neq j \end{cases} = \sum \sigma^2 = k\sigma^2$

Discrete Constant Velocity Process -1

- Formulation

$x(t + \Delta t) = x(t) + \Delta t\, v(t)$

$v(t + \Delta t) = v(t) + \underline{\eta} \qquad \eta \sim N(0, \Delta t \sigma_v^2)$, scalar representation

The velocity $v(t)$ wanders away from $v(0)$ with variance proportional to $t\sigma_v^2$

$$
\begin{bmatrix} x(t+\Delta t) \\ v(t+\Delta t) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \eta
$$

$= A(\Delta t) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \eta \qquad \eta \sim N(0, Q(\Delta t))$, matrix representation

$Q(\Delta t) = \begin{bmatrix} 0 & 0 \\ 0 & \Delta t \sigma_v^2 \end{bmatrix}$

Random vector

$X = \begin{bmatrix} x \\ v \end{bmatrix} \sim N(\mu, \Sigma) = N\left(\begin{bmatrix} \mu_x \\ \mu_v \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xv} \\ \Sigma_{vx} & \Sigma_{vv} \end{bmatrix}\right)$

$\mu \rightarrow E[A(\Delta t)X + \eta] = A(\Delta t)E[X] + E[\eta] = A(\Delta t)\mu = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_x \\ \mu_v \end{bmatrix}$

Propagating through one step

$= \begin{bmatrix} \mu_x \\ \mu_v \end{bmatrix} + \Delta t \begin{bmatrix} \mu_v \\ 0 \end{bmatrix} = \mu + \Delta t \begin{bmatrix} \mu_v \\ 0 \end{bmatrix}$

Discrete Constant Velocity Process -2

$$\Sigma \rightarrow \quad E[(AX+\eta)(AX+\eta)^T] - E[AX+\eta]E[AX+\eta]^T$$
$$= E[(AX+\eta)(X^TA^T+\eta^T)] - (AE[X]+E[\eta])(E[X]^TA^T+E[\eta]^T)$$
$$= AE[XX^T]A^T + AE[X\eta^T] + E[\eta X^T]A^T + E[\eta\eta^T]$$
$$-AE[X]E[X]^TA^T - AE[X]E[\eta]^T - E[\eta]E[X]^TA^T - E[\eta]E[\eta]^T$$
$$= A(E[XX^T] - E[X]E[X]^T)A^T + E[\eta\eta^T]$$
$$= A(\Delta t)\Sigma A(\Delta t)^T + Q(\Delta t)$$
$$= \begin{bmatrix}1 & \Delta t\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Sigma_{xx} & \Sigma_{xv}\\ \Sigma_{vx} & \Sigma_{vv}\end{bmatrix}\begin{bmatrix}1 & 0\\ \Delta t & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \Delta t\sigma_v^2\end{bmatrix}$$
$$= \begin{bmatrix}\Sigma_{xx} & \Sigma_{xv}\\ \Sigma_{vx} & \Sigma_{vv}\end{bmatrix} + \Delta t\begin{bmatrix}2\Sigma_{xv}+\Delta t\Sigma_{vv} & \Sigma_{vv}\\ \Sigma_{vv} & \sigma_v^2\end{bmatrix}$$
$$= \Sigma + \Delta t\begin{bmatrix}2\Sigma_{xv}+\Delta t\Sigma_{vv} & \Sigma_{vv}\\ \Sigma_{vv} & \sigma_v^2\end{bmatrix}$$

Function of Δt; one iteration of Δt $\neq$ two iterations of $\frac{\Delta t}{2}$

Reformulate the process in the continuous time domain

$$\Rightarrow \dot{\mu} = \begin{bmatrix}\mu_v\\ 0\end{bmatrix} \qquad \dot{\Sigma} = \begin{bmatrix}2\Sigma_{xv} & \Sigma_{vv}\\ \Sigma_{vv} & \sigma_v^2\end{bmatrix}$$

Discrete Constant Velocity Process -3

- Continuous constant velocity process

Integration…

$$u(t) = \begin{bmatrix}\mu_x + t\mu_v\\ \mu_v\end{bmatrix}$$

$$\Sigma(t) = \begin{bmatrix}\Sigma_{xx}+2t\Sigma_{xv}+t^2\Sigma_{vv}+\frac{t^3}{3}\sigma_v^2 & \Sigma_{xv}+t\Sigma_{vv}+\frac{t^2}{2}\sigma_v^2\\ \Sigma_{xv}+t\Sigma_{vv}+\frac{t^2}{2}\sigma_v^2 & \Sigma_{vv}+t\sigma_v^2\end{bmatrix}$$

$$\mu \rightarrow A(t)\mu = \mu + \begin{bmatrix}t\mu_v\\ 0\end{bmatrix}$$

Mean update is identical
in the discrete and continuous cases

$$\Sigma \rightarrow A(t)\Sigma A(t)^T + Q(t) = \Sigma + \begin{bmatrix}2t\Sigma_{xv}+t^2\Sigma_{vv} & t\Sigma_{vv}\\ t\Sigma_{vv} & 0\end{bmatrix} + \sigma_v^2\begin{bmatrix}\frac{t^3}{3} & \frac{t^2}{2}\\ \frac{t^2}{2} & t\end{bmatrix}$$

$$\Rightarrow Q(t) = \sigma_v^2\begin{bmatrix}\frac{t^3}{3} & \frac{t^2}{2}\\ \frac{t^2}{2} & t\end{bmatrix}$$

Process noise

Discrete Constant Acceleration Process -1

- Formulation

$$\begin{pmatrix} x \\ v \\ a \end{pmatrix} \sim N(\mu, \Sigma) = N\left(\begin{bmatrix} \mu_x \\ \mu_v \\ \mu_a \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xv} & \Sigma_{xa} \\ \Sigma_{vx} & \Sigma_{vv} & \Sigma_{va} \\ \Sigma_{ax} & \Sigma_{av} & \Sigma_{aa} \end{bmatrix}\right)$$

$$A(\Delta t) = \begin{bmatrix} 1 & \Delta t & 0 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} + O(\Delta t^2)$$

$$Q(\Delta t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t\sigma_a^2 \end{bmatrix}$$

$$\mu \to A(\Delta t)\mu = \mu + \Delta t \begin{bmatrix} \mu_v \\ \mu_a \\ 0 \end{bmatrix} + O(\Delta t^2)$$

$$\Sigma \to A(\Delta t)\Sigma A(\Delta t)^T + Q(t) = \Sigma + \Delta t \begin{bmatrix} 2\Sigma_{xv} & \Sigma_{vv} + \Sigma_{xa} & \Sigma_{va} \\ \Sigma_{vv} + \Sigma_{xa} & 2\Sigma_{va} & \Sigma_{aa} \\ \Sigma_{va} & \Sigma_{aa} & \sigma_a^2 \end{bmatrix} + O(\Delta t^2)$$

Discrete Constant Acceleration Process -2

Differentiation…

$$\dot{\mu} = \begin{bmatrix} \mu_v \\ \mu_a \\ 0 \end{bmatrix}$$

$$\dot{\Sigma} = \begin{bmatrix} 2\Sigma_{xv} & \Sigma_{vv} + \Sigma_{xa} & \Sigma_{va} \\ \Sigma_{vv} + \Sigma_{xa} & 2\Sigma_{va} & \Sigma_{aa} \\ \Sigma_{va} & \Sigma_{aa} & \sigma_a^2 \end{bmatrix}$$

$$\begin{aligned}
\Sigma_{a,a}(t) &= \Sigma_{a,a} + t \cdot \sigma_a^2 \\
\Sigma_{v,a}(t) &= \Sigma_{v,a} + t \cdot \Sigma_{a,a} + \frac{t^2}{2} \cdot \sigma_a^2 \\
\Sigma_{v,v}(t) &= \Sigma_{v,v} + 2t \cdot \Sigma_{v,a} + t^2 \cdot \Sigma_{a,a} + \frac{t^3}{3} \cdot \sigma_a^2 \\
\Sigma_{x,a}(t) &= \Sigma_{x,a} + t \cdot \Sigma_{v,a} + \frac{t^2}{2} \cdot \Sigma_{a,a} + \frac{t^3}{6} \cdot \sigma_a^2 \\
\Sigma_{x,v}(t) &= \Sigma_{x,v} + t \cdot (\Sigma_{v,v} + \Sigma_{x,a}) + \frac{3t^2}{2} \cdot \Sigma_{v,a} + \frac{t^3}{2} \cdot \Sigma_{a,a} + \frac{t^4}{8} \cdot \sigma_a^2 \\
\Sigma_{x,x}(t) &= \Sigma_{x,x} + 2t \cdot \Sigma_{x,v} + t^2 \cdot (\Sigma_{v,v} + \Sigma_{x,a}) + t^3 \cdot \Sigma_{v,a} + \frac{t^4}{4} \cdot \Sigma_{a,a} + \frac{t^5}{20} \cdot \sigma_a^2
\end{aligned}$$

Integration…

$$u(t) = \begin{bmatrix} \mu_x + t\mu_v + \frac{t^2}{2}\mu_a \\ t\mu_a \\ 0 \end{bmatrix}$$

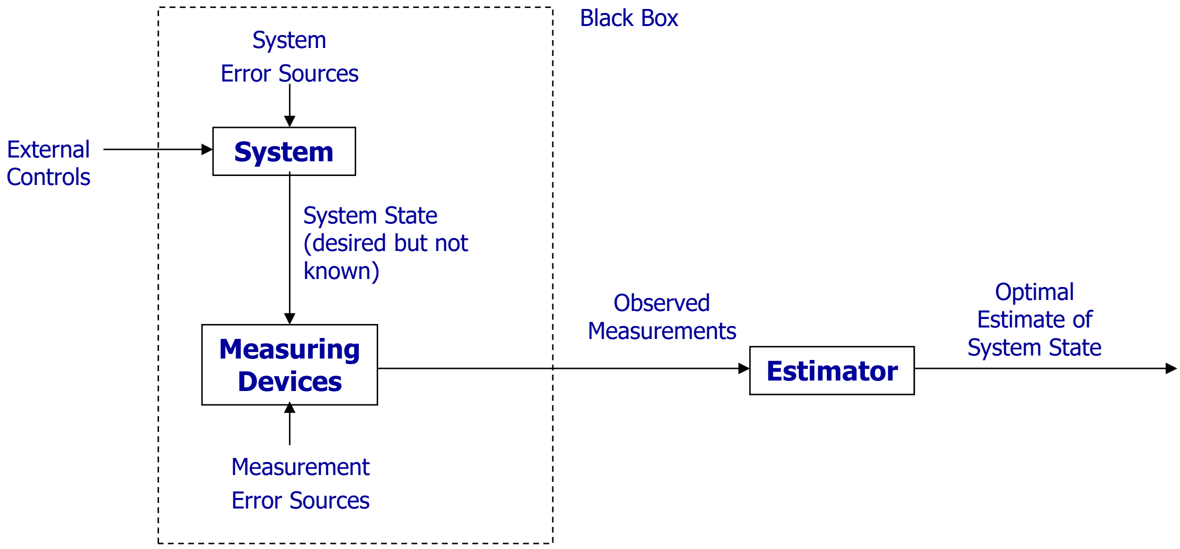
$$\Sigma(t) = \cdots$$

$$\Rightarrow \; Q(t) = \sigma_a^2 \begin{bmatrix} \frac{t^5}{20} & \frac{t^4}{8} & \frac{t^3}{6} \\ \frac{t^4}{8} & \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^3}{6} & \frac{t^2}{2} & t \end{bmatrix}$$

Process noise

The Problem

- Why do we need Kalman Filter?
  - "Usually" system states cannot be measured directly
  - Want to have "optimal" state estimation

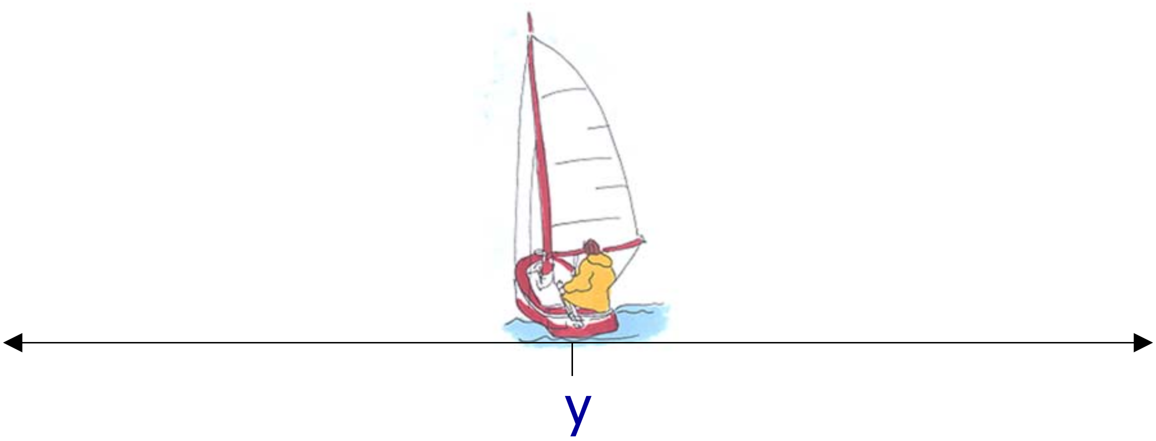


Kalman Filter

- Recursive data processing algorithm
  - Generates optimal estimate of desired quantities given the set of measurements
  - For linear system and white Gaussian errors, Kalman filter is "best" estimate based on all previous measurements
  - Recursive: Doesn't need to store all previous measurements and reprocess all data each time step

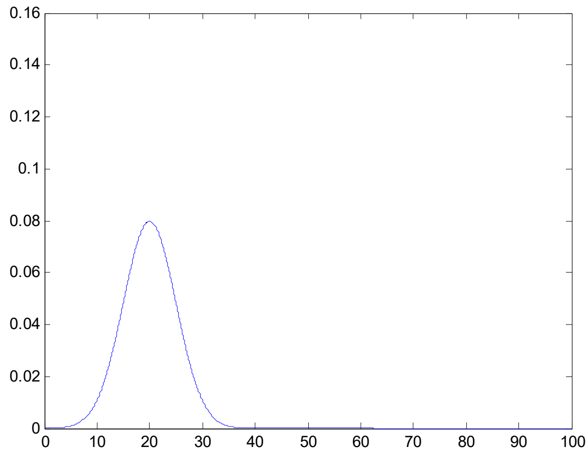
Conceptual Overview -1

- Example



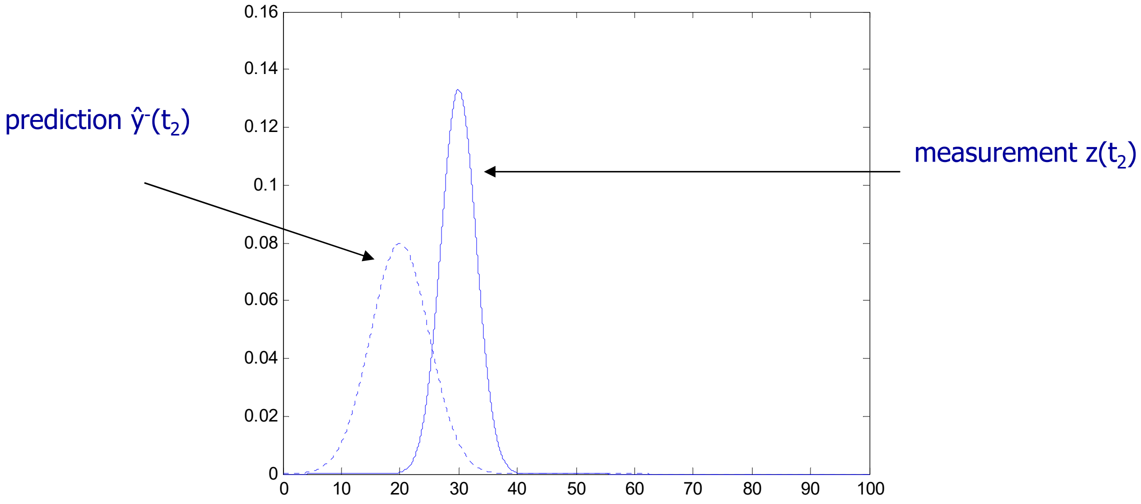
- Lost on the 1-dimensional line
- Position – y(t)
- Assume Gaussian distributed measurements

Conceptual Overview -2



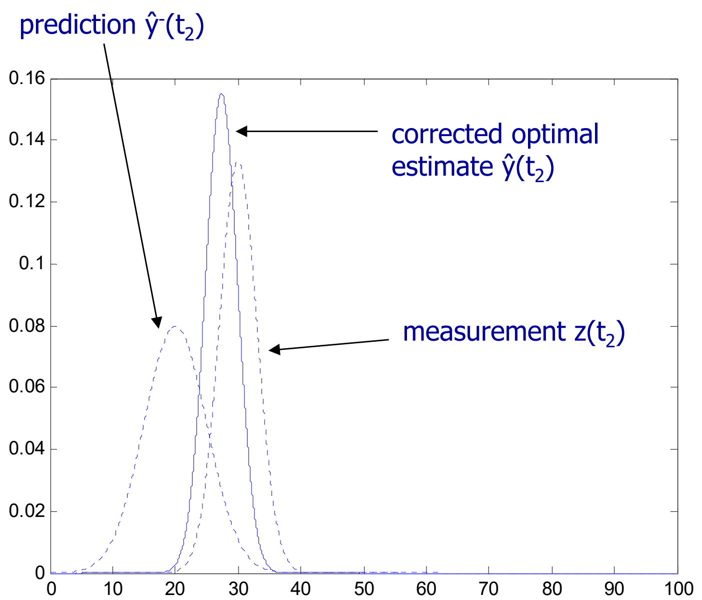
- Sextant Measurement at t_1: Mean = z_1 and Variance = σ_{z1}
  六分儀
- Optimal estimate of position is: $\hat{y}(t_1) = z_1$
- Variance of error in estimate: $\sigma^2_x(t_1) = \sigma^2_{z1}$
- Boat in same position at time t_2 - Predicted position is z_1

Conceptual Overview -3



- So we have the prediction $\hat{y}^-(t_2)$
- GPS Measurement at t_2: Mean = z_2 and Variance = σ_{z2}
- Need to correct the prediction due to measurement to get $\hat{y}(t_2)$
- Closer to more trusted measurement – linear interpolation?

Conceptual Overview -4



- Corrected mean is the new optimal estimate of position
- New variance is smaller than either of the previous two variances

Conceptual Overview -5

- Lesson so far

Make prediction based on previous data - $\hat{y}^-$, σ^-

↓

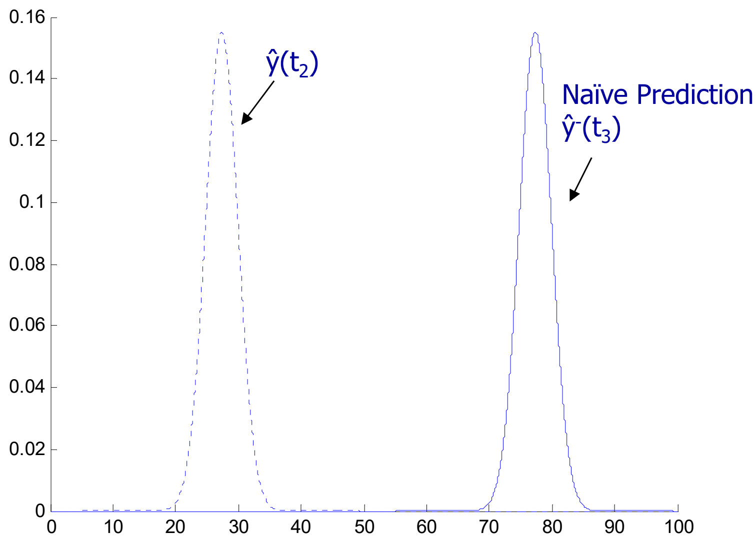
Take measurement – z_k, σ_z

↓

Optimal estimate ($\hat{y}$) = Prediction + (Kalman Gain) * (Measurement - Prediction)

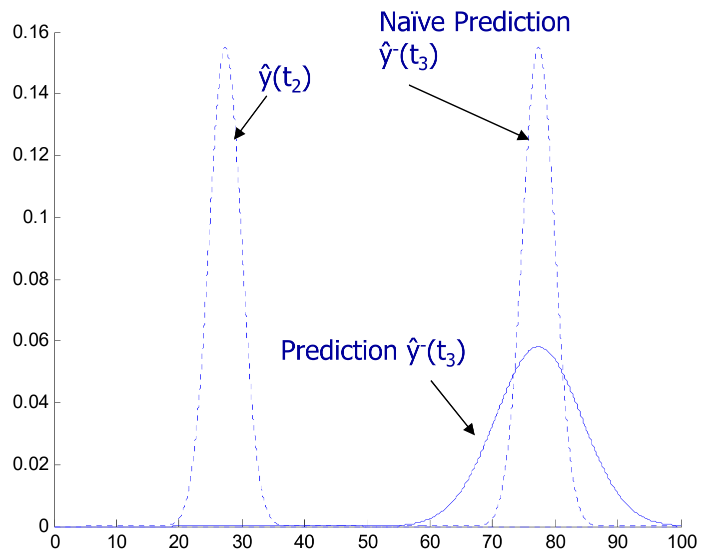
Variance of estimate = Variance of prediction * (1 – Kalman Gain)

Conceptual Overview -6



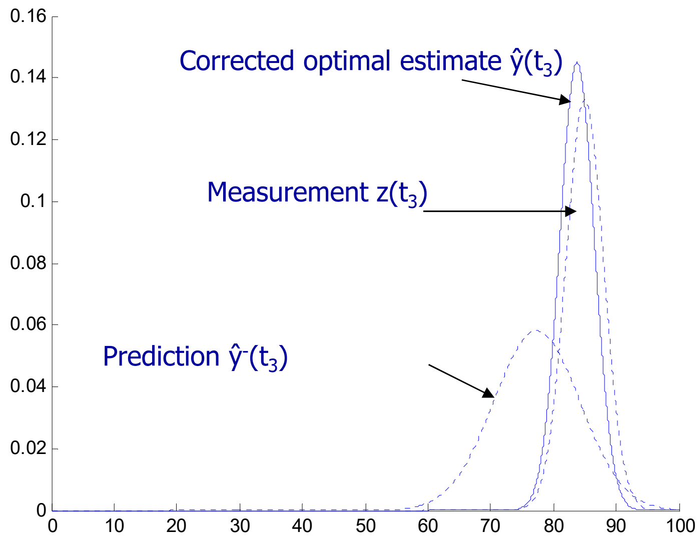
- At time t_3, boat moves with velocity dy/dt=u
- Naïve approach: Shift probability to the right to predict
- This would work if we knew the velocity exactly (perfect model)

Conceptual Overview -7



- Better to assume imperfect model by adding Gaussian noise
- dy/dt = u + w
- Distribution for prediction moves and spreads out

Conceptual Overview -8



- Now we take a measurement at t_3
- Need to once again correct the prediction
- Same as before

Conceptual Overview -9

- The procedure
  - Initial conditions ($\hat{y}_{k-1}$ and σ_{k-1})
  - Prediction ($\hat{y}^-_k$, σ^-_k)
    - Use initial conditions and model (eg. constant velocity) to make prediction
  - Measurement (z_k)
    - Take measurement
  - Correction ($\hat{y}_k$, σ_k)
    - Use measurement to correct prediction by 'blending' prediction and residual – always a case of merging only two Gaussians
    - Optimal estimate with smaller variance

Theoretical Basis -1

- Process to be estimated

$y_k = Ay_{k-1} + Bu_k + w_{k-1}$ Process Noise (w) with covariance Q

$z_k = Hy_k + v_k$ Measurement Noise (v) with covariance R

- Kalman Filter
  - Predicted: $\hat{y}^-_k$ is estimate based on measurements at previous time-steps

$\hat{y}^-_k = Ay_{k-1} + Bu_k$ Predict (a priori) state estimation

$P^-_k = AP_{k-1}A^T + Q$ Predict (a priori) covariance

  - Corrected: $\hat{y}_k$ has additional information – the measurement at time k

$\hat{y}_k = \hat{y}^-_k + K(z_k - H\hat{y}^-_k)$

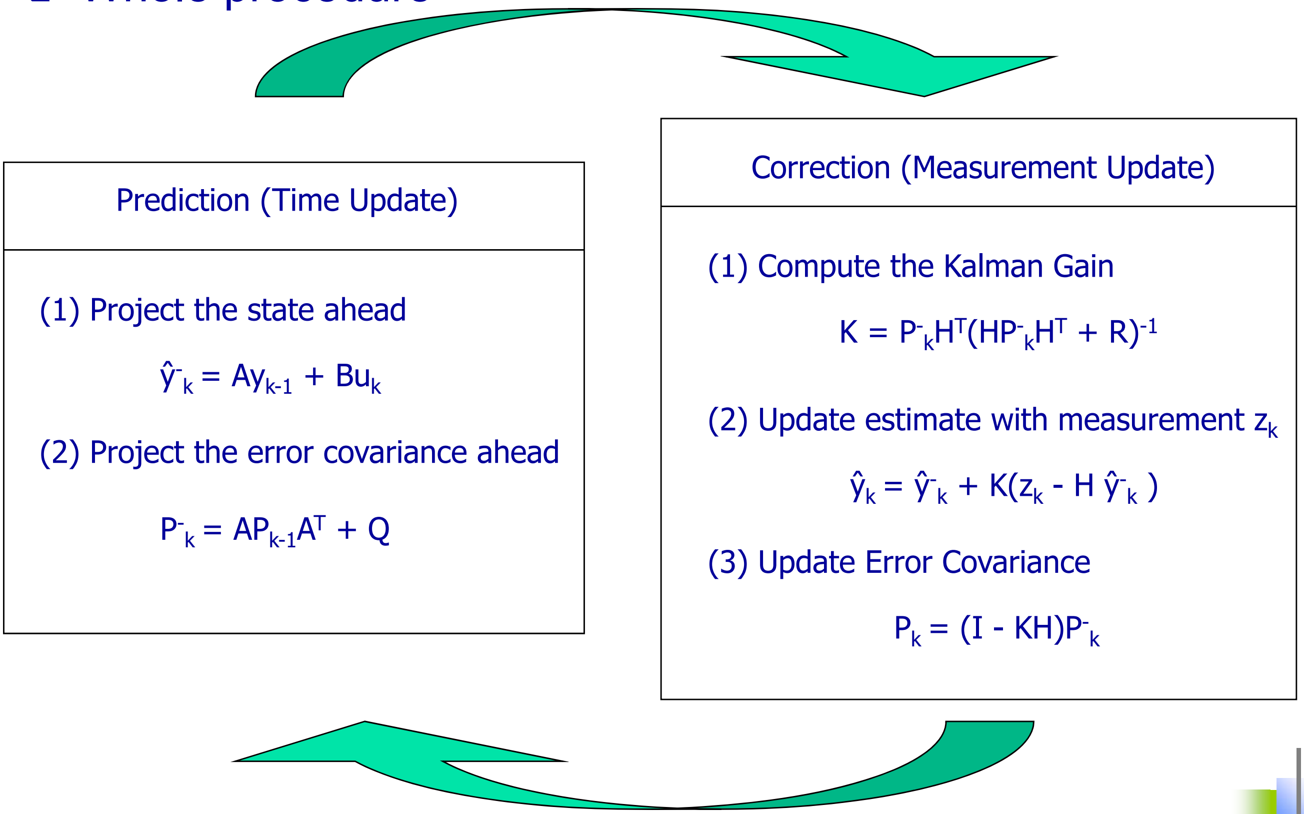
$P_k = (I - KH)P^-_k$

$K = P^-_k H^T (HP^-_k H^T + R)^{-1}$

Kalman gain

Theoretical Basis -2

- Whole procedure



Blending Factor

- If we are sure about measurements
  - Measurement error covariance (R) decreases to zero
  - K increases and weights residual more heavily than prediction

- If we are sure about prediction
  - Prediction error covariance P^-_k decreases to zero
  - K decreases and weights prediction more heavily than residual

Kalman Filter in Our Experiments -1

- System model

$$X_k = A_k X_{k-1} + B_k U_k + \omega_k \qquad \omega_k \sim N(0, Q_k)$$
$$Z_k = H_k X_k + v_k \qquad v_k \sim N(0, R_k)$$

- Time Update

$$X_k^- = A_k X_{k-1} + B_k U_k$$
$$P_k^- = A_k P_{k-1} A_k^T + Q_k$$

- Measurement update

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}$$
$$X_k = X_k^- + K_k (Z_k - H_k X_k^-)$$
$$P_k = P_k^- - K_k H_k P_k^-$$

Kalman Filter in Our Experiments -2

- Detailed models
  - The transition model (constant acceleration model)

$$A_k = \begin{bmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}$$

  - Case 1, acceleration feedback

$$H_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$K_k = \begin{bmatrix} \vdots & \vdots & \blacksquare \\ \vdots & \vdots & \blacksquare \\ \vdots & \vdots & \blacksquare \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \blacksquare \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + R_k \right)^{-1}$$

$$= \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} \frac{1}{P_{33} + R_k}$$

3x1, separately compensating disp., vel., and accel. states

Kalman Filter in Our Experiments -3

- Case 2, position feedback

$$H_k = [1 \quad 0 \quad 0] \qquad K_k = \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} \frac{1}{P_{11}+R_k}$$

- Case 3, position and acceleration feedback

$$H_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_k = \begin{bmatrix} \, & \vdots & \, \\ \, & \vdots & \, \\ \, & \vdots & \, \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \, & \cdots & \, \\ \cdots & \cdots & \cdots \\ \, & \cdots & \, \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} R_{k1} & 0 \\ 0 & R_{k2} \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} P_{11} & P_{13} \\ P_{21} & P_{23} \\ P_{31} & P_{33} \end{bmatrix} \left(\begin{bmatrix} P_{11}+R_{k1} & P_{13} \\ P_{31} & P_{33}+R_{k2} \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} P_{11} & P_{13} \\ P_{21} & P_{23} \\ P_{31} & P_{33} \end{bmatrix} \frac{\begin{bmatrix} P_{33}+R_{k2} & -P_{13} \\ -P_{31} & P_{11}+R_{k1} \end{bmatrix}}{(P_{11}+R_{k1})(P_{33}+R_{k2})-P_{13}P_{31}}$$

Kalman Filter in Our Experiments -4

- Case 4, position and velocity feedback

$$H_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The End

- Questions?