機器人簡介（ME5118）

Chap 6: Manipulator Dynamics

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Acceleration of a Rigid Body -1

- Differentiation of a velocity vector V_Q

$${}^{B}A_Q = \frac{d}{dt}\,{}^{B}V_Q = \lim_{\Delta t \to 0} \frac{{}^{B}V_Q(t+\Delta t) - {}^{B}V_Q(t)}{\Delta t}$$

Derivative of velocity vector ${}^{B}V_Q$ relative to frame $\{B\}$

$${}^{A}({}^{B}A_Q) = {}^{A}(\frac{d}{dt}\,{}^{B}V_Q)$$

Expressed in frame $\{A\}$

$$= {}^{A}_{B}R\,\underline{{}^{B}({}^{B}A_Q)} = {}^{A}_{B}R\,\underline{{}^{B}A_Q}$$

When both frames are the same

- - - - - - - - - - - - - - - - -

$$a_C = {}^{U}A_{C\,ORG}$$

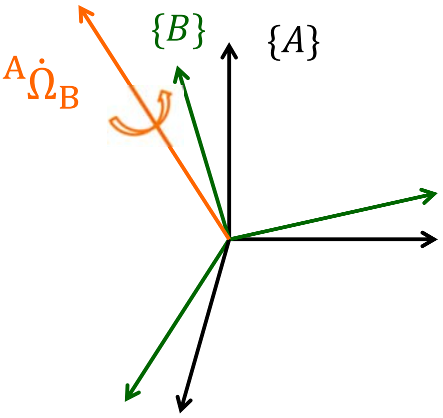
Acceleration of the origin of frame $\{C\}$ relative to the universe reference frame $\{U\}$

Acceleration of a Rigid Body -2

- Angular acceleration vector ${}^{A}\dot{\Omega}_{B}$

$$ {}^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,{}^{A}\Omega_{B} = \lim_{\Delta t\to 0}\frac{{}^{A}\Omega_{B}(t+\Delta t)-{}^{A}\Omega_{B}(t)}{\Delta t} $$

Derivative of angular velocity of frame $\{B\}$ relative to frame $\{A\}$



$$ {}^{C}({}^{A}\dot{\Omega}_{B}) $$

Expressed in frame $\{C\}$

$$ \dot{\omega}_{C} = {}^{U}\dot{\Omega}_{C} $$

Angular acceleration of frame $\{C\}$ relative to the universe reference frame $\{U\}$

Acceleration of a Rigid Body -3

- Angular acceleration

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

$$ \Big\downarrow \text{diff.} $$

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}\,{}^{A}_{B}R\,{}^{B}\Omega_{C} $$

$$ = {}^{A}\dot{\Omega}_{B} + {}^{A}_{B}R\,{}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

Rigid Body Motion -1

- Freshman Dynamics

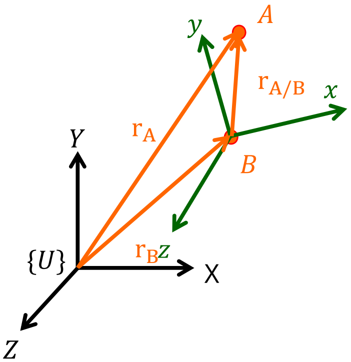
Following the materials described in Chap 5

$$\overrightarrow{v_A} = \overrightarrow{v_B} + \overrightarrow{v_{rel}} + \overrightarrow{\omega} \times \overrightarrow{r_{A/B}}$$

$$\overrightarrow{v_A} = (\dot{x}_B \hat{I} + \dot{y}_B \hat{J}) + (\dot{x}_{A/B} \hat{\imath} + \dot{y}_{A/B} \hat{\jmath}) + \overrightarrow{\omega} \times (x_{A/B} \hat{I} + y_{A/B} \hat{J})$$

$\Big\downarrow$ diff.

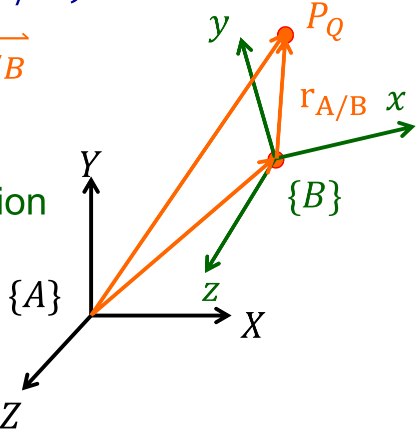
$$\begin{aligned}\overrightarrow{a_A} = {} & (\ddot{x}_B \hat{I} + \ddot{y}_B \hat{J}) \\ & + (\ddot{x}_{A/B} \hat{\imath} + \ddot{y}_{A/B} \hat{\jmath}) + \overrightarrow{\omega} \times (\dot{x}_{A/B} \hat{\imath} + \dot{y}_{A/B} \hat{\jmath}) \\ & + \dot{\overrightarrow{\omega}} \times (x_{A/B} \hat{I} + y_{A/B} \hat{J}) \\ & + \overrightarrow{\omega} \times ((\dot{x}_{A/B} \hat{\imath} + \dot{y}_{A/B} \hat{\jmath}) + \overrightarrow{\omega} \times (x_{A/B} \hat{I} + y_{A/B} \hat{J}))\end{aligned}$$



Rigid Body Motion -2

$$\begin{aligned}\overrightarrow{a_A} = {} & (\ddot{x}_B \hat{I} + \ddot{y}_B \hat{J}) \\ & + \dot{\overrightarrow{\omega}} \times (x_{A/B} \hat{I} + y_{A/B} \hat{J}) + \overrightarrow{\omega} \times (\overrightarrow{\omega} \times (x_{A/B} \hat{I} + y_{A/B} \hat{J})) \\ & + 2\overrightarrow{\omega} \times (\dot{x}_{A/B} \hat{\imath} + \dot{y}_{A/B} \hat{\jmath}) + (\ddot{x}_{A/B} \hat{\imath} + \ddot{y}_{A/B} \hat{\jmath})\end{aligned}$$

➡
$$\begin{aligned}\overrightarrow{a_A} = {} & \overrightarrow{a_B} + \dot{\overrightarrow{\omega}} \times \overrightarrow{r_{A/B}} + \overrightarrow{\omega} \times \overrightarrow{\omega} \times \overrightarrow{r_{A/B}} \\ & + \underbrace{2\overrightarrow{\omega} \times \overrightarrow{v_{rel}}}_{\text{Coriolis acceleration}} + \underbrace{\overrightarrow{a_{rel}}}_{\text{“relative” acceleration}}\end{aligned}$$



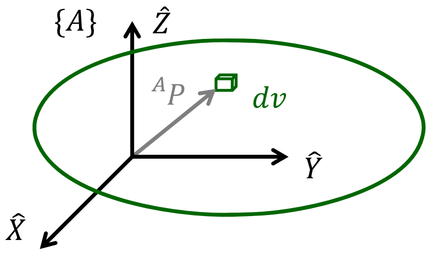
- Thus

$$\begin{aligned}{}^{A}A_{Q} = {} & {}^{A}A_{B\,ORG} \\ & + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\, {}^{B}P_{Q} + {}^{A}\Omega_{B} \times \left({}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}P_{Q}\right) \\ & + 2\, {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}V_{Q} + {}^{A}_{B}R\, {}^{B}A_{Q}\end{aligned}$$

Mass Distribution -1

- Inertia tensor relative to frame $\{A\}$

$$ {}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} $$



Mass moment of inertia >0

$$ I_{xx} = \iiint_V (y^2 + z^2)\rho dv $$

$$ I_{yy} = \iiint_V (x^2 + z^2)\rho dv $$

$$ I_{zz} = \iiint_V (x^2 + y^2)\rho dv $$

Mass product of inertia

$$ I_{xy} = \iiint_V xy\rho dv $$

$$ I_{xz} = \iiint_V xz\rho dv $$

$$ I_{yz} = \iiint_V yz\rho dv $$

Mass Distribution -2

- Inertia tensor
  - Constant real symmetric matrix (orthogonally diagonalizable)

    its eigendecomposition $(\mathrm{i.e.}, M = V\Lambda V^{-1})$

$$ {}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = R \begin{bmatrix} I_{XX} & 0 & 0 \\ 0 & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{bmatrix} R^T $$

principal moment of inertia

Rotation matrix,
revealing directions of the principal axes

  - $I_{xx} + I_{yy} + I_{zz} = trace\left({}^{A}I\right) = constant$
    - Trace is invariant under a similarity transformation
  - If xy-plane is plane of symmetry, then $I_{xz} = I_{yz} = 0$

Mass Distribution -3

- Parallel-axis Theorem
  - Computing how the inertia tensor changes under translations of the reference coordinate system

$$^{A}I_{zz} = {}^{C}I_{zz} + m({x_c}^2 + {y_c}^2)$$

$$^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$$

C: at COM of the body
A: arbitrary frame

Vector-matrix form

$$^{A}I = {}^{C}I + m[{P_c}^T P_c I_3 - P_c {P_c}^T]$$

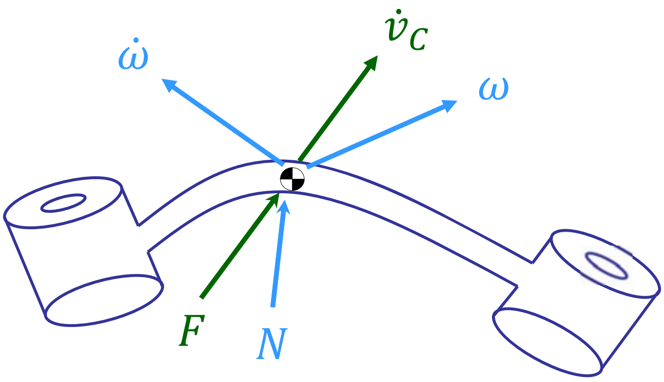
$$P_c = [x_c \quad y_c \quad z_c]^T$$

COM relative to $\{A\}$

Newton's Equation and Euler's Equation

- Newton's equation

$$F = \frac{d}{dt}(m v_C) = m \dot{v}_C$$



- Euler's equation

$$N = \frac{d}{dt}(\underline{I}\omega)$$

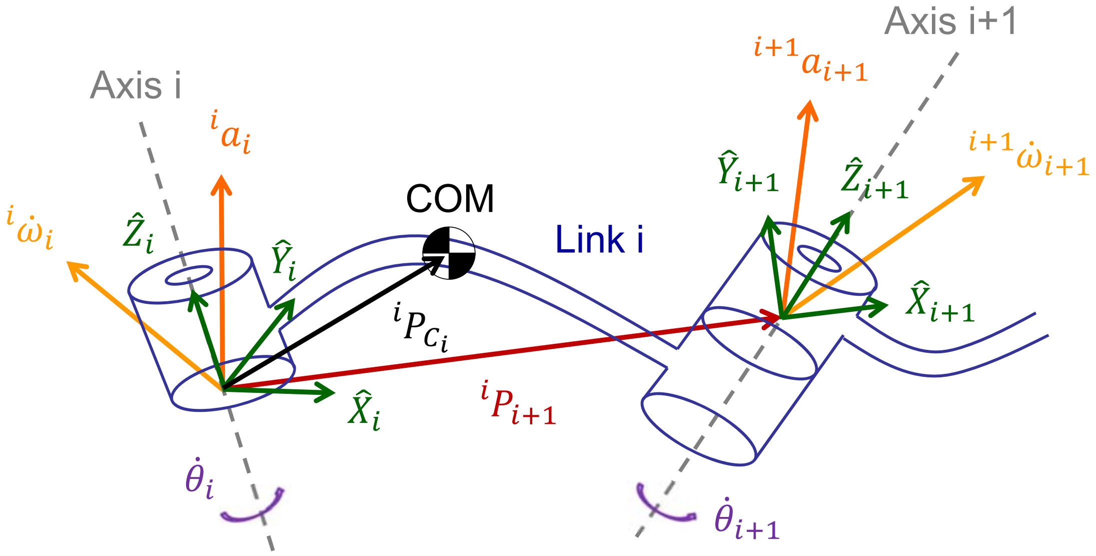
Even if using inertial frame, it can change during motion

$$N = \underline{{}^{C}I}\dot{\omega} + \omega \times {}^{C}I\omega$$

C: body frame, whose origin is located at COM
^{C}I: constant matrix

Acceleration “Propagation” from Link to Link -1

- Strategy: Represent linear and angular accelerations of link i in frame $\{i\}$, and find their relationship to those of neighboring links



Acceleration “Propagation” from Link to Link -2

- Rotational Joint (Link i+1)
  - Angular acceleration propagation

In Chap 5

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\,\underline{\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

$$ \Big\downarrow \text{diff.} \qquad \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ {}^{i}\dot{\omega}_{i+1} = {}^{i}\dot{\omega}_{i} + {}^{i}\omega_{i} \times {}^{i}_{i+1}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + {}^{i}_{i+1}R\,\ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ \Big\downarrow {}^{i+1}_{i}R $$

$$ {}^{i+i}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\dot{\omega}_{i} + {}^{i+1}_{i}R\ {}^{i}\omega_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

Acceleration “Propagation” from Link to Link -3

- Linear acceleration propagation

$$ {}^{i}a_{i+1} = {}^{i}a_{i} + {}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) $$

$$ \Big\downarrow \; {}^{i+1}_{\;\;i}R $$

$$ {}^{i+1}a_{i+1} = {}^{i+1}_{\;\;i}R \left({}^{i}a_{i} + {}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) \right) $$

Acceleration “Propagation” from Link to Link -4

- ❑ Prismatic joint (Link i+1)
  - Angular acceleration propagation

$$ {}^{i}\dot{\omega}_{i+1} = {}^{i}\dot{\omega}_{i} \xrightarrow{{}^{i+1}_{\;\;i}R} {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\;\;i}R \; {}^{i}\dot{\omega}_{i} $$

  - Linear acceleration propagation

$$ {}^{i}a_{i+1} = {}^{i}a_{i} + {}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) $$
$$ +2\, {}^{i}\omega_{i} \times {}_{i+1}^{\;\;i}R \dot{d}_{i+1} {}^{i+1}\hat{Z}_{i+1} + {}_{i+1}^{\;\;i}R \underline{\ddot{d}_{i+1} {}^{i+1}\hat{Z}_{i+1}} $$

$$ \Big\downarrow \; {}^{i+1}_{\;\;i}R \qquad\qquad \ddot{d}_{i+1} {}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}a_{i+1} = {}^{i+1}_{\;\;i}R \left({}^{i}a_{i} + {}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) \right) $$
$$ +2\, {}^{i+1}\omega_{i+1} \times \dot{d}_{i+1} {}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1} {}^{i+1}\hat{Z}_{i+1} $$

Acceleration “Propagation” from Link to Link -5

- COM

$$ {}^{i}a_{C_i} = {}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{C_i} \right) $$

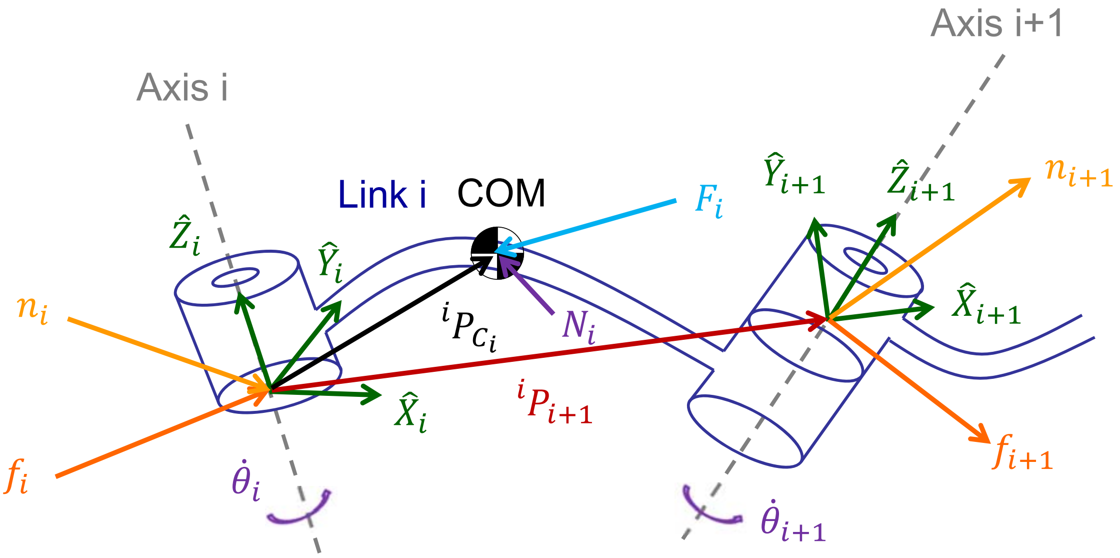
C_i: COM of the ith link

Force Propagation from Link to Link -1

- Inertia force and torque acting at the COM

$$ F_i = m a_{C_i} $$

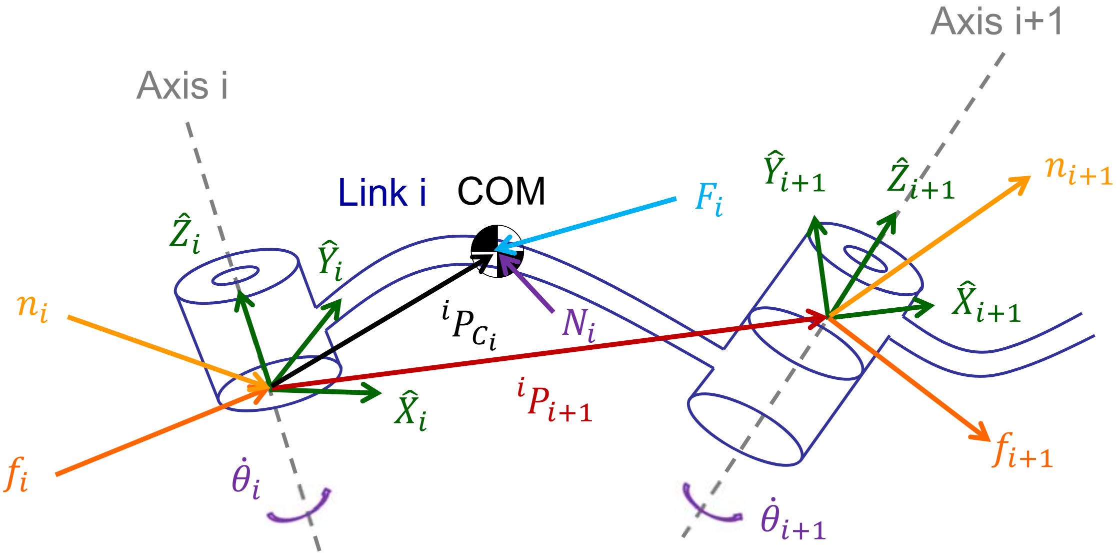
$$ N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i $$



Force Propagation from Link to Link -2

- $$^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

  $$^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}N_i + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$



Force Propagation from Link to Link -3

- Thus
  - Revolute joint

    $$\tau_i = {}^{i}n_i^T\ {}^{i}\hat{Z}_i$$

  - Prismatic joint

    $$\tau_i = {}^{i}f_i^T\ {}^{i}\hat{Z}_i$$

- Comments
  - Inclusion of gravity force: $^{0}a_0 = g = 9.81\ m/s$
  - A manipulator moving in free space: $^{N+1}f_{N+1} = 0\ \ {}^{N+1}n_{N+1} = 0$

Iterative Newton-Euler Dynamic Formulation

- Outward iterations
  - Link 1 to link n
  - Velocities and accelerations
- Inward iterations
  - Link n to link 1
  - Forces and torques

- Revolute joint vs. prismatic joint: Choose correct equations
- General structure, can be applied to any manipulator
- Easy for numerical computation

Example: A RR Manipulator -1

- Conditions:

$$ {}^{1}P_{C_1} = l_1\hat{X}_1 \qquad {}^{C_1}I_1 = 0 $$

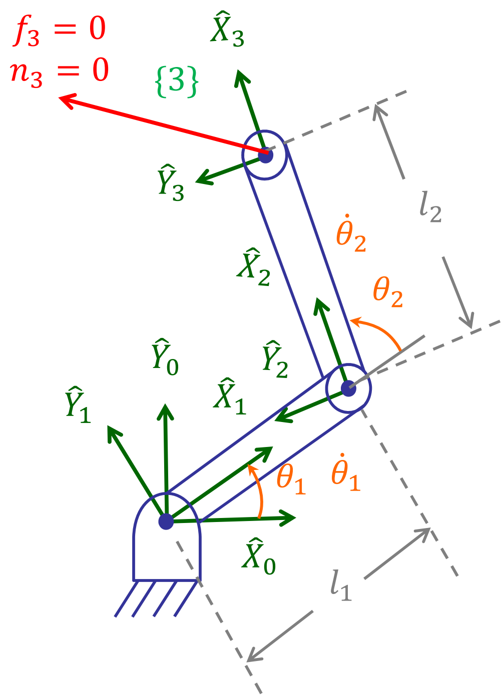
$$ {}^{2}P_{C_2} = l_2\hat{X}_2 \qquad {}^{C_2}I_2 = 0 $$

$$ m_1, m_2 $$

$$ \omega_0 = 0 \qquad {}^{0}\dot{v}_0 = g\hat{Y}_0 $$

$$ \dot{\omega}_0 = 0 $$

$$ {}_{i+1}^{\;\;i}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Example: A RR Manipulator -2

- Velocity and acceleration propagations

$${}^1\omega_1 = {}^1_0R\,{}^0\cancel{\omega_0} + \dot{\theta}_1\,{}^1\hat{Z}_1 = \dot{\theta}_1\,{}^1\hat{Z}_1 = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}$$

$${}^1\dot{\omega}_1 = {}^1_0R\,{}^0\cancel{\dot{\omega}_0} + {}^1_0R\,{}^0\cancel{\omega_0} \times \dot{\theta}_1\,{}^1\hat{Z}_1 + \ddot{\theta}_1\hat{Z}_1 = \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix}$$

$${}^1a_1 = {}^1_0R(\,{}^0a_0 + \,{}^0\cancel{\dot{\omega}_0} \times \,{}^0\cancel{P_1} + \,{}^0\cancel{\omega_0} \times (\,{}^0\cancel{\omega_0} \times \,{}^0\cancel{P_1})) = \begin{bmatrix}c_1 & s_1 & 0\\-s_1 & c_1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\g\\0\end{bmatrix}$$

$${}^1a_{C_1} = \,{}^1a_1 + \,{}^1\dot{\omega}_1 \times \,{}^1P_{C_1} + \,{}^1\omega_1 \times (\,{}^1\omega_1 \times \,{}^1P_{C_1}) \qquad = \begin{bmatrix}gs_1\\gc_1\\0\end{bmatrix}$$

$$= \begin{bmatrix}gs_1\\gc_1\\0\end{bmatrix} + \begin{bmatrix}0\\l_1\ddot{\theta}_1\\0\end{bmatrix} + \begin{bmatrix}-l_1\dot{\theta}_1{}^2\\0\\0\end{bmatrix} = \begin{bmatrix}-l_1\dot{\theta}_1{}^2 + gs_1\\l_1\ddot{\theta}_1 + gc_1\\0\end{bmatrix}$$

Example: A RR Manipulator -3

$${}^1F_1 = m\,{}^1a_{C_1}\begin{bmatrix}-m_1l_1\dot{\theta}_1{}^2 + m_1gs_1\\m_1l_1\ddot{\theta}_1 + m_1gc_1\\0\end{bmatrix}$$

$${}^1N_1 = {}^{C_1}\cancel{I}\,{}^1\dot{\omega}_1 + \,{}^1\omega_1 \times {}^{C_1}\cancel{I}\,{}^1\omega_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$${}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \dot{\theta}_2\,{}^2\hat{Z}_2 = \begin{bmatrix}0\\0\\\dot{\theta}_1 + \dot{\theta}_2\end{bmatrix}$$

$${}^2\dot{\omega}_2 = {}^2_1R\,{}^1\dot{\omega}_1 + {}^2_1R\,{}^1\omega_1 \cancel{\times} \dot{\theta}_2\,{}^2\hat{Z}_2 + \ddot{\theta}_2\,{}^2\hat{Z}_2 = \begin{bmatrix}0\\0\\\ddot{\theta}_1 + \ddot{\theta}_2\end{bmatrix}$$

$${}^2a_2 = {}^2_1R(\,{}^1a_1 + \,{}^1\dot{\omega}_1 \times \,{}^1P_2 + \,{}^1\omega_1 \times (\,{}^1\omega_1 \times \,{}^1P_2)) =$$

$$= \begin{bmatrix}c_2 & s_2 & 0\\-s_2 & c_2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-l_1\dot{\theta}_1{}^2 + gs_1\\l_1\ddot{\theta}_1 + gc_1\\0\end{bmatrix} = \begin{bmatrix}l_1\ddot{\theta}_1s_2 - l_1\,\dot{\theta}_1{}^2c_2 + gs_{12}\\l_1\ddot{\theta}_1c_2 + l_1\,\dot{\theta}_1{}^2s_2 + gc_{12}\\0\end{bmatrix}$$

Example: A RR Manipulator -4

$$^{2}a_{C_2} = {}^{2}a_2 + {}^{2}\dot{\omega}_2 \times {}^{2}P_{C_2} + {}^{2}\omega_2 \times \left({}^{2}\omega_2 \times {}^{2}P_{C_2} \right)$$

$$= \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1 \dot{\theta}_1^{\,2} c_2 + g s_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1 \dot{\theta}_1^{\,2} s_2 + g c_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix}$$

$$^{2}F_2 = m\, {}^{2}a_{C_2} = \begin{bmatrix} m_2 l_1 \ddot{\theta}_1 s_2 - m_2 l_1 \dot{\theta}_1^{\,2} c_2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 \ddot{\theta}_1 c_2 + m_2 l_1 \dot{\theta}_1^{\,2} s_2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$^{2}N_2 = {}^{C_2}I\, {}^{2}\dot{\omega}_2 + {}^{2}\omega_2 \times {}^{C_2}I\, {}^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: A RR Manipulator -5

- Force and torque propagations

$$^{2}f_2 = {}^{2}_{3}R\, {}^{3}f_3 + {}^{2}F_2 = {}^{2}F_2$$

$$^{2}n_2 = {}^{2}_{3}R\, {}^{3}\cancel{n_3} + {}^{2}\cancel{N_2} + {}^{2}P_{C_2} \times {}^{2}F_2 + {}^{2}P_3 \times {}^{2}_{3}R\, {}^{3}\cancel{f_3}$$

$$= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^{\,2} + m_2 l_2 g c_{12} + m_2 l_2^{\,2} (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

$$^{1}f_1 = {}^{1}_{2}R\, {}^{2}f_2 + {}^{1}F_1$$

$$= \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^{\,2} + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^{\,2} + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^{\,2} + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}$$

Example: A RR Manipulator -6

$$
\begin{aligned}
{}^{1}n_1 &= {}^{1}_{2}R\,{}^{2}n_2 + \cancel{{}^{1}N_1} + {}^{1}P_{C_1} \times {}^{1}F_1 + {}^{1}P_2 \times {}^{1}_{2}R\,{}^{2}f_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 {\dot{\theta}_1}^{2} + m_2 l_2 g c_{12} + m_2 {l_2}^{2} (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ 0 \\ m_1 {l_1}^{2} \ddot{\theta}_1 + m_1 l_1 g c_1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ 0 \\ m_2 {l_1}^{2} \ddot{\theta}_1 - m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^{2} + m_2 l_1 g s_2 s_{12} + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}
\end{aligned}
$$

Example: A RR Manipulator -7

- Joint torques

$$
\begin{aligned}
\tau_1 &= {}^{1}n_1^{T}\,{}^{1}\widehat{Z_1} \\
&= m_2 {l_2}^{2} (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) + (\mathrm{m}_1 + \mathrm{m}_2) {l_1}^{2} \ddot{\theta}_1 \\
&\quad - m_2 l_1 l_2 s_2 {\dot{\theta}_2}^{2} - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1
\end{aligned}
$$

$$
\begin{aligned}
\tau_2 &= {}^{2}n_2^{T}\,{}^{2}\widehat{Z_2} \\
&= m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 {\dot{\theta}_1}^{2} + m_2 l_2 g c_{12} + m_2 {l_2}^{2} (\ddot{\theta}_1 + \ddot{\theta}_2)
\end{aligned}
$$

The Structure of Dynamic Equations -1

- The state-space equation

$$\underset{n\times 1}{\tau} = \underset{n\times n}{M(\Theta)}\underset{n\times 1}{\ddot{\Theta}} + \underset{n\times 1}{V\left(\Theta,\dot{\Theta}\right)} + \underset{n\times 1}{G(\Theta)}$$

Mass matrix

Centrifugal Coriolis

gravity

- Revisit the RR manipulator

$$M(\Theta) = \begin{bmatrix} l_2{}^2 m_2 + 2l_1 l_2 m_2 c_2 + l_1{}^2(m_1+m_2) & l_2{}^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2{}^2 m_2 + l_1 l_2 m_2 c_2 & l_2{}^2 m_2 \end{bmatrix}$$

$$V\left(\Theta,\dot{\Theta}\right) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2{}^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1{}^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1+m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

The Structure of Dynamic Equations -2

- The configuration-space equation

$$\underset{n\times 1}{\tau} = \underset{n\times n}{M(\Theta)}\ddot{\Theta} + \underset{n\times\frac{n(n-1)}{2}}{B(\Theta)}\left[\dot{\Theta}\dot{\Theta}\right] + \underset{n\times n}{C(\Theta)}\left[\dot{\Theta}^2\right] + \underset{n\times 1}{G(\Theta)}$$

Mass matrix

Coriolis

Centrifugal

gravity

$n\times 1$

$$\underset{\frac{n(n-1)}{2}\times 1}{\left[\dot{\Theta}\dot{\Theta}\right]} = \left[\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad \dots \quad \dot{\theta}_{n-1}\dot{\theta}_n\right]^T$$

$$\underset{n\times 1}{\left[\dot{\Theta}^2\right]} = \left[\dot{\theta}_1{}^2 \quad \dot{\theta}_2{}^2 \quad \dots \quad \dot{\theta}_n{}^2\right]^T$$

The Structure of Dynamic Equations -3

- Revisit the RR manipulator

$$V(\Theta,\dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^{\,2} - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^{\,2} \end{bmatrix} = B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2]$$

$$B(\Theta) = \begin{bmatrix} -2m_2 l_1 l_2 s_2 \\ 0 \end{bmatrix} \qquad [\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1 \dot{\theta}_2]$$

$$C(\Theta) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 \end{bmatrix} \qquad [\dot{\Theta}^2] = \begin{bmatrix} \dot{\theta}_1^{\,2} \\ \dot{\theta}_2^{\,2} \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics -1

- Newton-Euler: Force-moment-based analysis

  Lagrange: Energy-based analysis

- Of course, for a system, both methods should yield the same equations of motion

Lagrangian Formulation of Manipulator Dynamics -2

- Kinetic energy

$$k_i = \frac{1}{2} m_i v_{C_i}^T v_{C_i} + \frac{1}{2}\ {}^i\omega_i^T\, {}^{C_i}I_i\ {}^i\omega_i$$

$$k = \sum_{i=1}^{n} k_i \qquad k = k(\Theta, \dot{\Theta}) = \frac{1}{2}\dot{\Theta}^T \mathrm{M}(\Theta)\dot{\Theta}$$

- Potential energy

$$u_i = -m_i\ {}^0g^T\ {}^0P_{C_i} + \underline{u_{ref_i}}$$

Shift the zero reference height

$$u = \sum_{i=1}^{n} u_i \qquad u = u(\Theta)$$

Lagrangian Formulation of Manipulator Dynamics -3

- Lagrangian

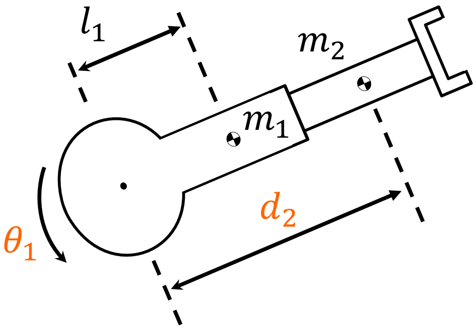
$$\mathcal{L}(\Theta, \dot{\Theta}) = k(\Theta, \dot{\Theta}) - u(\Theta)$$

- Equation of motion for the manipulator

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} - \frac{\partial \mathcal{L}}{\partial \Theta} = \tau$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau$$

Example: An RP Manipulator -1



$$^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}$$

$$^{C_2}I_2 = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}$$

- Kinetic energy

$$k_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz1} \dot{\theta}_1^2$$

$$k_2 = \frac{1}{2} m_2 (d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2) + \frac{1}{2} I_{zz2} \dot{\theta}_1^2$$

$$k(\Theta, \dot{\Theta}) = \frac{1}{2} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2) \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{d}_2^2$$

Example: An RP Manipulator -2

- Potential energy

$$u_1 = m_1 g l_1 sin\theta_1 + m_1 g l_1$$

$$u_2 = m_2 g d_2 sin\theta_1 + m_2 g d_{2max}$$

$$u(\Theta) = (m_1 l_1 + m_2 d_2) g sin\theta_1 + \underline{m_1 g l_1 + m_2 g d_{2max}}$$

Shift the zero reference height

- Lagrangian

$$\frac{\partial k}{\partial \dot{\Theta}} = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2) \dot{\theta}_1 \\ m_2 \dot{d}_2 \end{bmatrix}$$

$$\frac{\partial k}{\partial \Theta} = \begin{bmatrix} 0 \\ m_2 d_2 \ \dot{\theta}_1^2 \end{bmatrix}$$

$$\frac{\partial u}{\partial \Theta} = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g cos\theta_1 \\ m_2 g \sin\theta_1 \end{bmatrix}$$

Example: An RP Manipulator -3

- Equations of motion

$$\tau_1 = (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ +(m_1 l_1 + m_2 d_2) g cos\theta_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g sin\theta_1$$

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state-space representation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

$$M(\Theta) = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g cos\theta_1 \\ m_2 g sin\theta_1 \end{bmatrix}$$

Manipulator Dynamics in Cartesian Space -1

- Dynamic equations
  - In joint space

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

  - In Cartesian space

$$F = M_x(\Theta)\ddot{X} + V_x\left(\Theta, \dot{\Theta}\right) + G_x(\Theta)$$

- Formulation

$$\tau = J^T(\Theta)F$$

$$F = J^{-T}\tau = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}V\left(\Theta, \dot{\Theta}\right) + J^{-T}G(\Theta)$$

$$\dot{X} = J\dot{\Theta} \qquad \ddot{X} = \dot{J}\dot{\Theta} + J\ddot{\Theta} \qquad \ddot{\Theta} = J^{-1}\ddot{X} - J^{-1}\dot{J}\dot{\Theta}$$

Manipulator Dynamics in Cartesian Space -2

$$F = J^{-T}M(\Theta)J^{-1}\ddot{X} - J^{-T}M(\Theta)J^{-1}\dot{J}\dot{\Theta} + J^{-T}V\left(\Theta,\dot{\Theta}\right) + J^{-T}G(\Theta)$$
$$= M_x(\Theta)\ddot{X} + V_x\left(\Theta,\dot{\Theta}\right) + G_x(\Theta)$$

$$M_x(\Theta) = J^{-T}(\Theta)M(\Theta)J^{-1}(\Theta)$$

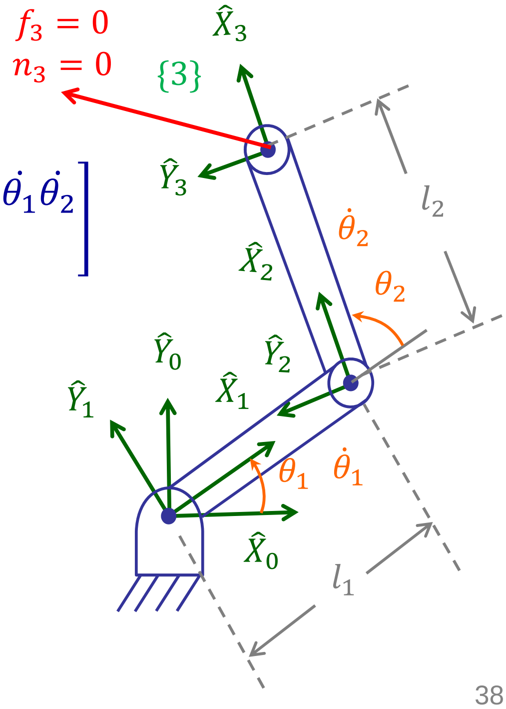
$$V_x\left(\Theta,\dot{\Theta}\right) = J^{-T}(\Theta)\left(V\left(\Theta,\dot{\Theta}\right) - M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$$

$$G_x(\Theta) = J^{-T}(\Theta)G(\Theta)$$

Revisit Example: A RR Manipulator -1

- In joint space

$$M(\Theta) = \begin{bmatrix} {l_2}^2 m_2 + 2l_1 l_2 m_2 c_2 + {l_1}^2(m_1+m_2) & {l_2}^2 m_2 + l_1 l_2 m_2 c_2 \\ {l_2}^2 m_2 + l_1 l_2 m_2 c_2 & {l_2}^2 m_2 \end{bmatrix}$$



$$V\left(\Theta,\dot{\Theta}\right) = \begin{bmatrix} -m_2 l_1 l_2 s_2 {\dot{\theta}_2}^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 {\dot{\theta}_1}^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1+m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

Revisit Example: A RR Manipulator -2

- Jacobian

$$J(\Theta)=\begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \qquad J^{-1}(\Theta)=\frac{1}{l_1 l_2 s_2}\begin{bmatrix} l_2 & 0 \\ -l_1 c_2 - l_2 & l_1 s_2 \end{bmatrix}$$

$$\dot{J}(\Theta)=\begin{bmatrix} l_1 c_2 \dot{\theta}_2 & 0 \\ -l_1 s_2 \theta_2 & 0 \end{bmatrix}$$

- In Cartesian space

$$M_x(\Theta)=J^{-T}(\Theta)M(\Theta)J^{-1}(\Theta)=\begin{bmatrix} m_2+\frac{m_1}{{s_2}^2} & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V_x(\Theta,\dot{\Theta})=J^{-T}(\Theta)\left(V(\Theta,\dot{\Theta})-M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$$

$$=\begin{bmatrix} -(m_2 l_1 c_2 + m_2 l_2){\dot{\theta}_1}^{\ 2} - m_2 l_2 {\dot{\theta}_2}^{\ 2} - (2m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \frac{c_2}{{s_2}^2})\dot{\theta}_1\dot{\theta}_2 \\ m_2 l_1 s_2 {\dot{\theta}_1}^{\ 2} + m_2 l_1 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

$$G_x(\Theta)=J^{-T}(\Theta)G(\Theta)=\begin{bmatrix} m_1 g\frac{c_1}{s_2}+m_2 g s_{12} \\ m_2 g c_{12} \end{bmatrix}$$

Torque Equation

- In Cartesian space

$$\tau = J^T(\Theta)F = J^T(\Theta)(M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta))$$

$$\tau = J^T(\Theta)M_x(\Theta)\ddot{X} + B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\Theta}^2] + G(\Theta)$$

➡ $J^T(\Theta)V_x(\Theta,\dot{\Theta}) = B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\Theta}^2]$

Revisit Example: A RR Manipulator

$$J^T(\Theta)V_x(\Theta,\dot{\Theta}) = B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\theta}^2]$$

$$= \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} -(m_2 l_1 c_2 + m_2 l_2)\dot{\theta}_1{}^2 - m_2 l_2 \dot{\theta}_2{}^2 - (2m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \frac{c_2}{s_2{}^2})\dot{\theta}_1\dot{\theta}_2 \\ m_2 l_1 s_2 \dot{\theta}_1{}^2 + l_1 m_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

$$B_x(\Theta) = \begin{bmatrix} -m_1 l_1{}^2 \frac{c_2}{s_2} - m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 \end{bmatrix}$$

$$C_x(\Theta) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 \end{bmatrix}$$

Friction

- Viscous friction

$$\tau_{friction} = c\dot{\theta}$$

- Coulomb friction

$$\tau_{friction} = c\ sgn\dot{\theta}$$

$\dot{\theta} = 0$, $c =$ “static coefficient”

$\dot{\theta} \neq 0$, $c =$ “dynamic coefficient”

The End

- Questions?