Chap 4 Inverse Manipulator Kinematics

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Let’s start here…

- Forward kinematics of manipulators

Given θ_i in ${}^{w}_{i}T$, calculate ${}^{w}\{H\}$ or ${}^{w}P$

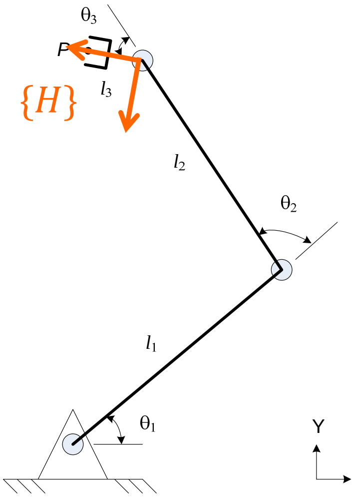
$${}^{w}_{H}T = f(\theta_1, \dots, \theta_i, \dots, \theta_n)$$

$${}^{w}P = {}^{w}_{i}T\ {}^{i}P\ , i = 1, \dots, n$$

- Inverse kinematics – This chapter

Given ${}^{w}\{H\}$ or ${}^{w}P$, calculate θ_i in ${}^{w}_{i}T$

$$[\theta_1, \dots, \theta_i, \dots, \theta_n] = f^{-1}({}^{w}_{H}T)$$

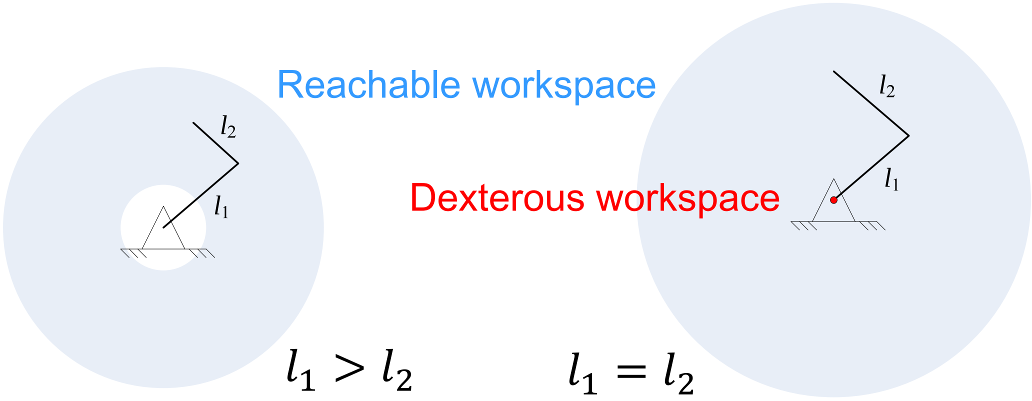


Solvability -1

- Assuming a robot has 6 DOFs
  - 6 unknown joint angles (or displacements) $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$
- Given ${}^6_0T$: 16 values
  - 9 from rotation matrix (6 constraints, only 3 independent equations)
  - 3 from translational displacement (3 independent equations)
  - 4 trivial (4th row: [0 0 0 1])
- Thus, 6 equations & 6 unknowns
  - Nonlinear transcendental equations

Solvability -2

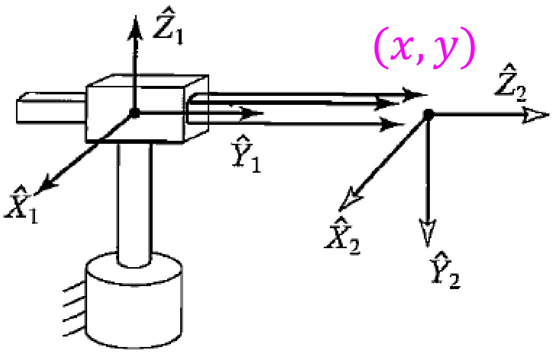
- Reachable workspace
  - The volume of space that the robot end-effector can reach in at least one orientation
- Dexterous workspace
  - The volume of space that the robot end-effector can reach with all orientations



$l_1 > l_2$ $l_1 = l_2$

Subspace

- Using a 2-DOF RP manipulator as an example
  - Variables (x, y)



$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad & \quad & {}^0\hat{Z}_2 & {}^0P_{2\,ORG} \end{array} \\
{}^0_2T = & \begin{bmatrix} \frac{y}{\sqrt{x^2+y^2}} & 0 & \boxed{\frac{x}{\sqrt{x^2+y^2}}} & \boxed{x} \\ \frac{-x}{\sqrt{x^2+y^2}} & 0 & \boxed{\frac{y}{\sqrt{x^2+y^2}}} & \boxed{y} \\ 0 & -1 & \boxed{0} & \boxed{0} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

Solvability -3

- Number of solutions
  - Depend on number of joints
  - Also depends on link parameters

    Ex: A manipulator with 6 rotational joints

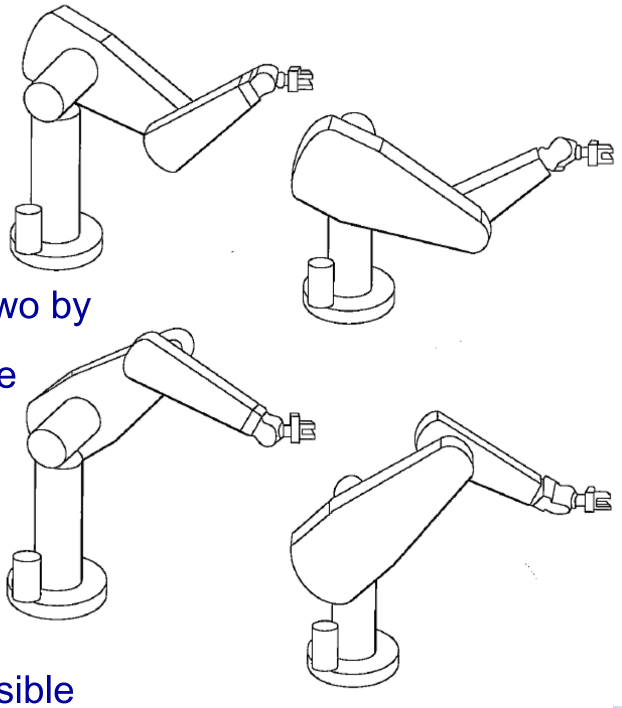
a_i	Number of solutions
$a_1 = a_3 = a_5 = 0$	$\leqslant 4$
$a_3 = a_5 = 0$	$\leqslant 8$
$a_3 = 0$	$\leqslant 16$
All $a_i \neq 0$	$\leqslant 16$

Solvability -4

- Ex: PUMA (6 rotational joints)
  - 8 different solutions
  - Four by variation of position structure

    Shown here
  - (For each position structure), two by variation of orientation structure

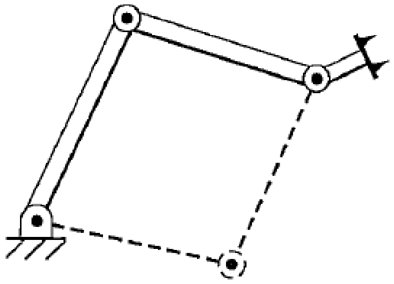
$$\theta_4' = \theta_4 + 180^\circ$$
$$\theta_5' = -\theta_5$$
$$\theta_6' = \theta_6 + 180^\circ$$



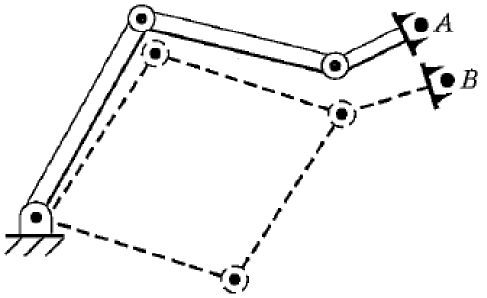
  - Some of them may be inaccessible

Solvability -5

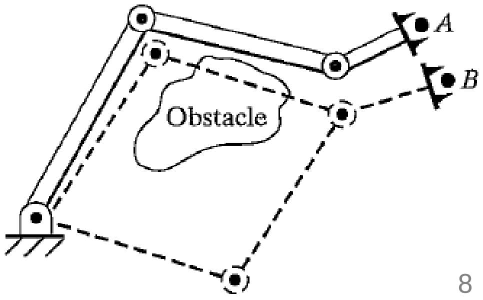
- When multiple solutions exist



  - Solution selection criteria
    - Closest solution



    - Avoid Obstacles



Solvability -6

- Method of solution
  - Solvable: the joint variables can be determined by an algorithm that allows one to determine ALL the sets of joint variables associated with a given position and orientation
    - Closed-form solutions – by algebraic method or geometric method
    - Numerical solutions
  - Now virtually all industrial manipulators has a closed-form solution
    - Sufficient condition: three neighboring joint axes intersect at a point

P is “necessary” for Q: Q can’t be true unless P is true

P is “sufficient” for Q: knowing P to be true is adequate to conclude that Q is true

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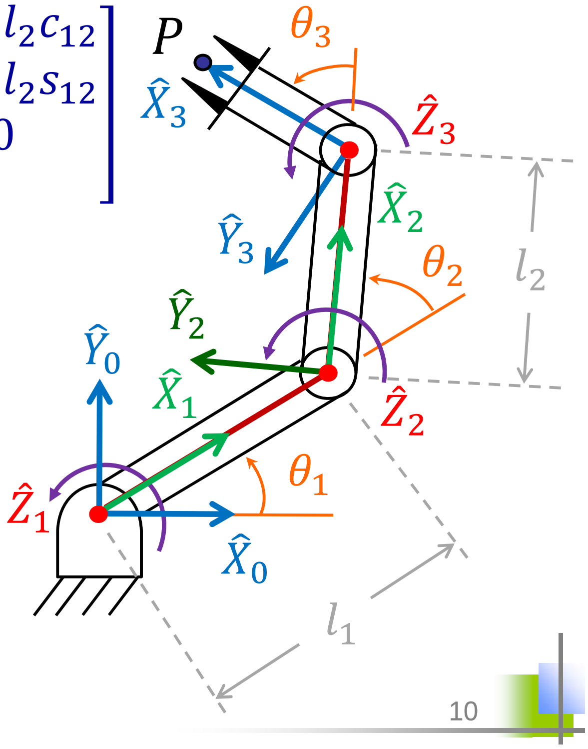
Example: A 3-DOF RRR Manipulator -1

- Inverse kinematics problem $(\theta_1, \theta_2, \theta_3) = ?$
  - Forward kinematics

$$ {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0.0 & l_1 s_1 + l_2 s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



  - Goal point

$$ {}^{0}_{3}T = \begin{bmatrix} c_{\phi} & -s_{\phi} & 0.0 & x \\ s_{\phi} & c_{\phi} & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Example: A 3-DOF RRR Manipulator -2

- Geometric solution: Decompose the spatial geometry into several plane-geometry problems

$$x^2+y^2=l_1^2+l_2^2-2l_1l_2\cos(180^\circ-\theta_2)$$

$$c_2=\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}$$

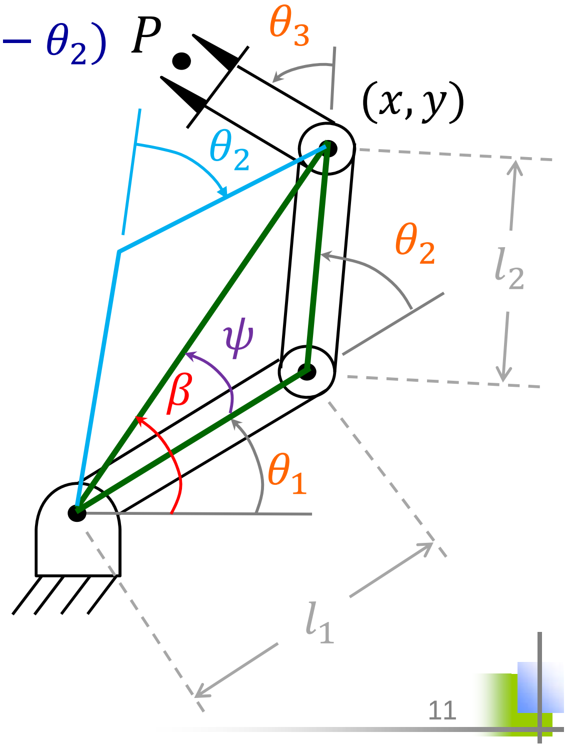
餘弦定理

$$cos\psi=\frac{x^2+y^2+l_1^2-l_2^2}{2l_1\sqrt{x^2+y^2}}$$

三角形內角 $0^\circ<\psi<180^\circ$

$$\theta_1=\begin{cases}atan2(y,x)+\psi & \theta_2<0^\circ\\ atan2(y,x)-\psi & \theta_2>0^\circ\end{cases}$$

$$\theta_3=\phi-\theta_1-\theta_2$$



Example: A 3-DOF RRR Manipulator -3

- Algebraic Solution
  - Setup equations

$$c_\phi=c_{123}$$
$$s_\phi=s_{123}$$
$$x=l_1c_1+l_2c_{12}$$
$$y=l_1s_1+l_2s_{12}$$

$$ {}^0_3T=\begin{bmatrix}c_{123} & -s_{123} & 0.0 & l_1c_1+l_2c_{12}\\ s_{123} & c_{123} & 0.0 & l_1s_1+l_2s_{12}\\ 0.0 & 0.0 & 1.0 & 0.0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}c_\phi & -s_\phi & 0.0 & x\\ s_\phi & c_\phi & 0.0 & y\\ 0.0 & 0.0 & 1.0 & 0.0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

  - Solve θ_2

$$x^2+y^2=l_1{}^2+l_2{}^2+2l_1l_2c_2$$

$$c_2=\frac{x^2+y^2-l_1{}^2-l_2{}^2}{2l_1l_2}$$

>1 or <1: too far for the manipulator to reach

⇨ within 1: “two solutions” $\theta_2=cos^{-1}(c_2)$

Example: A 3-DOF RRR Manipulator -4

- Put θ_2 back to the equations

$$x = (l_1 + l_2c_2)c_1 + (-l_2s_2)s_1 \triangleq k_1c_1 - k_2s_1$$
$$y = (l_1 + l_2c_2)s_1 + \quad (l_2s_2)c_1 \triangleq k_1s_1 + k_2c_1$$

- Change variables

define

$$r = +\sqrt{{k_1}^2 + {k_2}^2}$$
$$\gamma = Atan2(k_2, k_1)$$

then

$$k_1 = r\cos\gamma$$
$$k_2 = r\sin\gamma$$

And then

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1 = \cos(\gamma + \theta_1)$$
$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1 = \sin(\gamma + \theta_1)$$

Example: A 3-DOF RRR Manipulator -5

- Solve θ_1

$$\gamma + \theta_1 = Atan2\left(\frac{y}{r}, \frac{x}{r}\right) = Atan2(y, x)$$

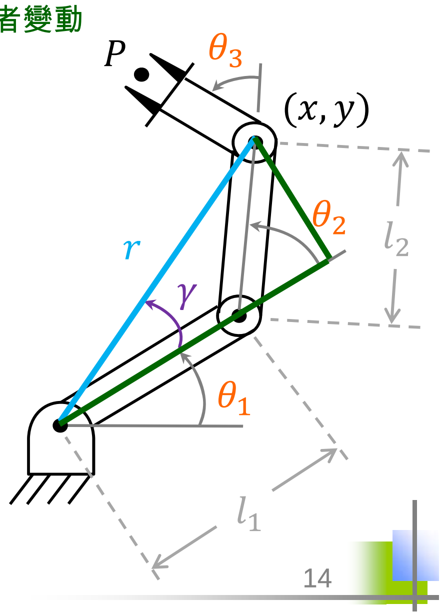
⇨ $$\theta_1 = Atan2(y, x) - Atan2(k_2, k_1)$$

當θ_2選不同解， $c2$和$s2$變動，k_1和k_2變動，θ_1也跟者變動

- Solve θ_3

$$\theta_1 + \theta_2 + \theta_3 = Atan2(s_\phi, c_\phi) = \phi$$

⇨ $$\theta_3 = \phi - \theta_1 - \theta_2$$



Solution by Reduction to Polynomial -1

- Transcendental equations (nonlinear)
  - Ex: How to solve θ of the following this equation?

$$acos\theta + \mathrm{b}sin\theta = c$$

  - Method: Change to polynomials

    Polynomials closed-form-solvable – up to 4 degree

$$\tan\left(\frac{\theta}{2}\right) = u, \qquad \cos\theta = \frac{1-u^2}{1+u^2}, \qquad \sin\theta = \frac{2u}{1+u^2}$$

Solution by Reduction to Polynomial -2

- Solution:

$$acos\theta + \mathrm{b}sin\theta = c$$

$$a\frac{1-u^2}{1+u^2} + \mathrm{b}\frac{2\mathrm{u}}{1+u^2} = c$$

$$(a+c)u^2 - 2bu + (c-a) = 0$$

$$u = \frac{\mathrm{b} \pm \sqrt{b^2 + a^2 - c^2}}{\mathrm{a+c}}$$

a, b, c大小有限制, 不一定有解

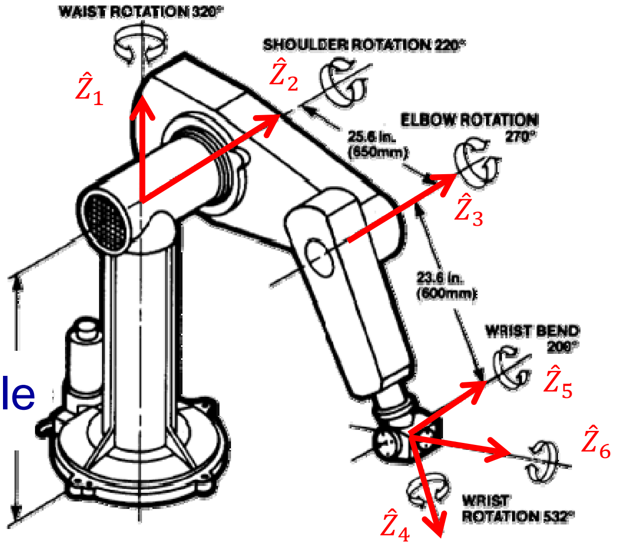
$$\theta = 2\tan^{-1}\left(\frac{\mathrm{b} \pm \sqrt{b^2 + a^2 - c^2}}{\mathrm{a+c}}\right)$$

If $a + c = 0$ $\Rightarrow$ $\theta = 180^{\circ}$

Pieper's Solution -1

- A 6-DOF manipulator which has three consecutive axes intersect at a point has a closed-form solution
- Usually,

  the last three joints intersect
  - First 3 DOFs – positioning
  - Last 3 DOFs – orientation
- Here, RRRRRR as an example
  - ${}^{0}P_{6\,ORG} = {}^{0}P_{4\,ORG}$



PUMA 560, image from www.iisartonline.org

Pieper's Solution -2

- Positioning structure
  - 法則：讓$\theta_1, \theta_2, \theta_3$層層分離

Note: $${}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}P_{4\,ORG} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\ {}^{3}P_{4\,ORG}$$

$$= {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

4th column of ${}^{3}_{4}T$

so

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{2}_{3}T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix}$$

讓$\theta_1, \theta_2, \theta_3$層層分離， f為θ_3函數

$$f_1(\theta_3) = a_3c_3 + d_4 s\alpha_3 s_3 + a_2$$

$$f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2$$

$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

Pieper's Solution -3

- Next

$$ {}^{0}P_{4ORG} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{0}_{1}T \begin{bmatrix} g_1(\theta_3) \\ g_2(\theta_3) \\ g_3(\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

讓$\theta_1, \theta_2, \theta_3$層層分離， g為θ_2, θ_3函數

$$ g_1(\theta_2, \theta_3) = c_2 f_1 - s_2 f_2 + a_1 $$
$$ g_2(\theta_2, \theta_3) = s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 $$
$$ g_3(\theta_2, \theta_3) = s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 $$

$$ r = x^2 + y^2 + z^2 = g_1^2 + g_2^2 + g_3^2 $$

r僅為θ_2, θ_3函數

$$ = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + 2a_1(c_2 f_1 - s_2 f_2) $$
$$ = (k_1 c_2 + k_2 s_2) 2a_1 + k_3 $$

$$ k_1(\theta_3) = f_1 $$
$$ k_2(\theta_3) = -f_2 $$
$$ k_3(\theta_3) = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 $$

Pieper's Solution -4

- In addition

$$ z = g_3 = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4 $$

z僅為θ_2, θ_3函數

$$ k_1(\theta_3) = f_1 $$
$$ k_2(\theta_3) = -f_2 $$
$$ k_4(\theta_3) = f_3 c\alpha_1 + d_2 c\alpha_1 $$

- Consider r and z together

$$ \begin{cases} r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3 \\ z = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4 \end{cases} $$

  - If $a_1 = 0$, $r = k_3(\theta_3) = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3$
  - If $s\alpha_1 = 0$, $z = k_4(\theta_3) = f_3 c\alpha_1 + d_2 c\alpha_1$
  - Else

$$ \frac{s\alpha_1 (r - k_3)^2}{4a_1^2} + \frac{(z - k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2 $$

Solve θ_3 of all three cases by using "$u = \tan\left(\frac{\theta_3}{2}\right)$"

Pieper's Solution -5

- Finally

  Using $r = (k_1 c_2 + k_2 s_2) 2 a_1 + k_3$ to solve θ_2

  Using $x = c_1 g_1(\theta_2, \theta_3) - s_1 g_2(\theta_2, \theta_3)$ to solve θ_1

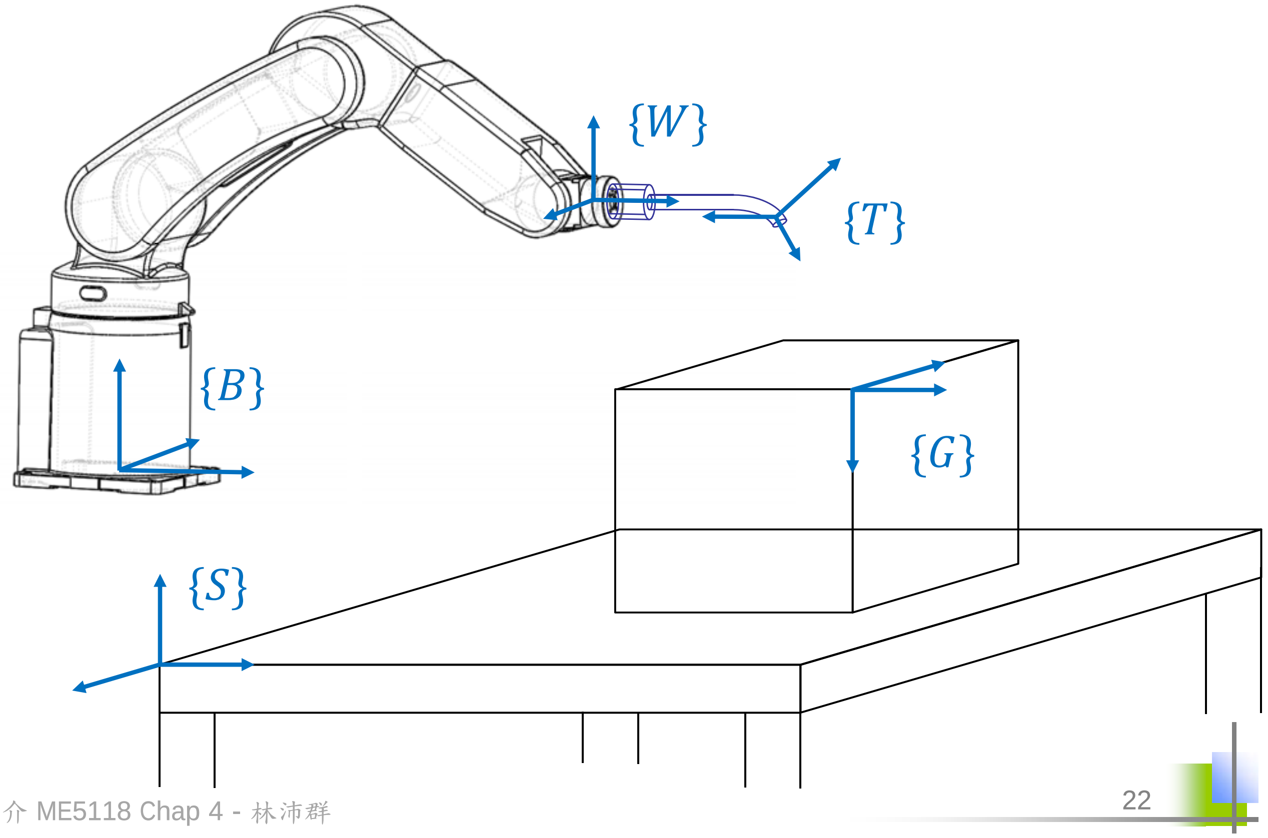
- Orientation
  - $\theta_1, \theta_2, \theta_3$ are known

    $${}^{3}_{6}R = {}^{0}_{3}R^{-1}\,{}^{0}_{6}R$$

  - Using Z-Y-Z Euler angle to solve $\theta_4, \theta_5, \theta_6$

The Standard Frames

- Base, wrist, tool, station, and goal frames



Technical terms

- Resolution (Repeatability)
  - How precise a manipulator can return to a given point
- Accuracy
  - The precision with which a computed point can be attained

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- Questions?