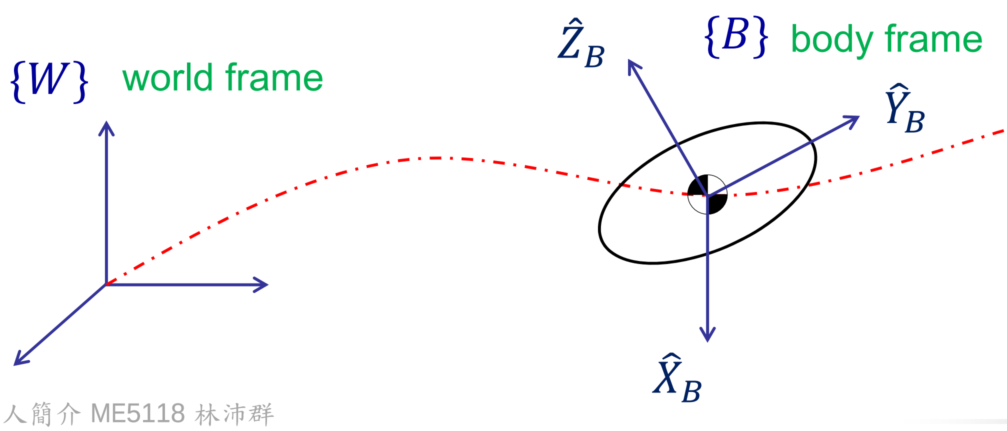
Chap 2 Spatial Description and Transformations

林沛群
國立台灣大學
機械工程學系

導讀

- 如何描述一個剛體(Rigid body)的運動狀態？
  - 平面：移動 2 DOFs、轉動 1 DOF（Degree of freedom）
  - 空間：移動 3 DOFs、轉動 3 DOFs
  - 各個DOF，具有displacement/orientation、velocity、acceleration等狀態描述
- 「建立frame」，以整合表達上列多個DOF的狀態



移動

$$P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

"column vector"

- A vector (i.e., displacement, frame basis)

  $\{B\}$: $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$ represented in $\{A\}$: $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$

- A position in space (i.e., position vector)

  ${}^{A}P_{B\,org}$ = origin of $\{B\}$ represented in $\{A\}$

- An order set of three numbers

轉動：Rotation Matrix -1

$${}^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix}$$

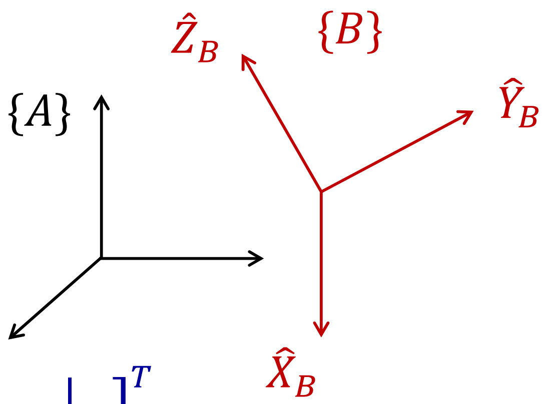
B relative to A

"column vector"

R的三個columns即為frame $\{B\}$的$basis$: $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$（由$\{A\}$看）

$$= \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

"direct cosines"



$$= \begin{bmatrix} - & {}^{B}\hat{X}_A{}^{T} & - \\ - & {}^{B}\hat{Y}_A{}^{T} & - \\ - & {}^{B}\hat{Z}_A{}^{T} & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^{B}\hat{X}_A & {}^{B}\hat{Y}_A & {}^{B}\hat{Z}_A \\ | & | & | \end{bmatrix}^{T} = {}^{B}_{A}R^{T}$$

轉動：Rotation Matrix -2

Moreover

$$ {}^A_BR^T\,{}^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}^T \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

$$ = \begin{bmatrix} - & {}^A\hat{X}_B{}^T & - \\ - & {}^A\hat{Y}_B{}^T & - \\ - & {}^A\hat{Z}_B{}^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

$$ = I_3 $$

3x3 identity matrix

$$ = {}^A_BR^{-1}\,{}^A_BR $$

$$ \Rightarrow {}^A_BR^T = {}^A_BR^{-1} = {}^B_AR $$

轉動：Rotation Matrix -3

以「point表達法」來解釋

orig coordinate $ {}^BP = {}^BP_x\hat{X}_B + {}^BP_y\hat{Y}_B + {}^BP_z\hat{Z}_B $

new coordinate $ {}^AP = {}^AP_x\hat{X}_A + {}^AP_y\hat{Y}_A + {}^AP_z\hat{Z}_A $

where $ {}^AP_x = {}^BP\cdot\hat{X}_A = \hat{X}_B\cdot\hat{X}_A\,{}^BP_x + \hat{Y}_B\cdot\hat{X}_A\,{}^BP_y + \hat{Z}_B\cdot\hat{X}_A\,{}^BP_z $

$$ {}^AP_y = {}^BP\cdot\hat{Y}_A = \hat{X}_B\cdot\hat{Y}_A\,{}^BP_x + \hat{Y}_B\cdot\hat{Y}_A\,{}^BP_y + \hat{Z}_B\cdot\hat{Y}_A\,{}^BP_z $$

$$ {}^AP_z = {}^BP\cdot\hat{Z}_A = \hat{X}_B\cdot\hat{Z}_A\,{}^BP_x + \hat{Y}_B\cdot\hat{Z}_A\,{}^BP_y + \hat{Z}_B\cdot\hat{Z}_A\,{}^BP_z $$

$$ \Rightarrow {}^AP = {}^A\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} \hat{X}_B\cdot\hat{X}_A & \hat{Y}_B\cdot\hat{X}_A & \hat{Z}_B\cdot\hat{X}_A \\ \hat{X}_B\cdot\hat{Y}_A & \hat{Y}_B\cdot\hat{Y}_A & \hat{Z}_B\cdot\hat{Y}_A \\ \hat{X}_B\cdot\hat{Z}_A & \hat{Y}_B\cdot\hat{Z}_A & \hat{Z}_B\cdot\hat{Z}_A \end{bmatrix} {}^B\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = {}^A_BR\,{}^BP $$

和p3結果相同

Rotation Matrix特性

- An orthogonal matrix $AA^T = I$
  - Always invertible $A^{-1} = A^T$
  - Columns: orthonormal basis
    - Length = 1
    - Mutually perpendicular
  - R有9個數字，但上列兩個條件置入6個constraints，所以R只有3個DOFs，與空間中轉動具有3 DOFs相符
  - Determinant =1 (rotation); =-1 (mirror)

Special Rotation Matrices -1

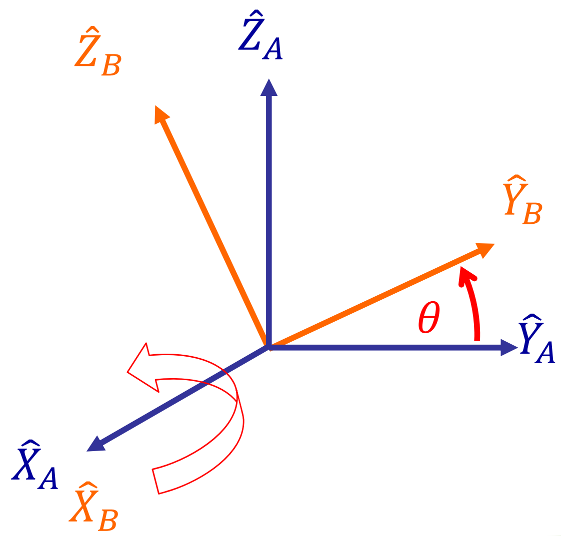
- About $\hat{X}_A$ with θ

旋轉角度

$$R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

旋轉軸

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

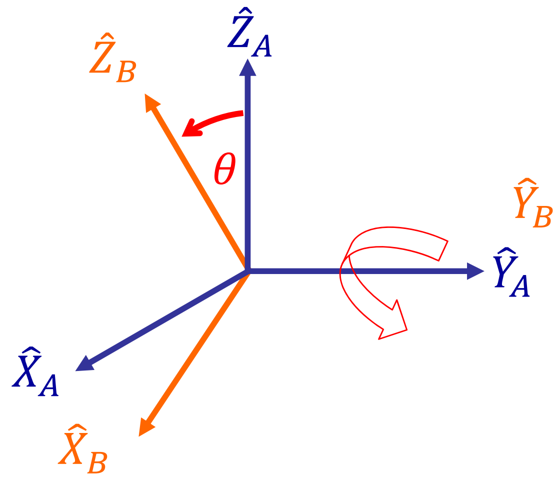


Special Rotation Matrices -2

- About $\hat{Y}_A$ with θ

$$R_{\hat{Y}_A}(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

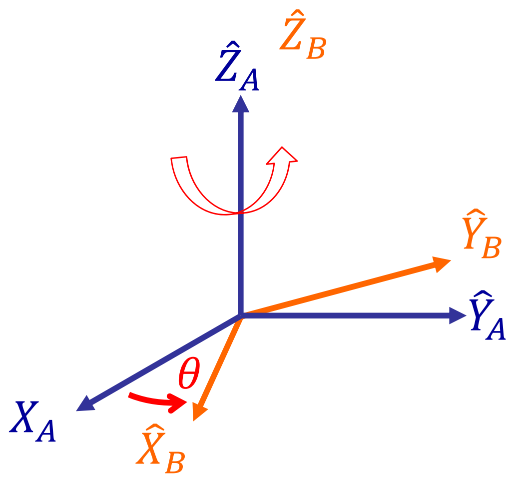


Special Rotation Matrices -3

- About $\hat{Z}_A$ with θ

$$R_{\hat{Z}_A}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Mapping -1

- Changing descriptions from frame to frame

- Frame description

$$\{B\} = \{{}^{A}_{B}R,\ {}^{A}P_{B\,org}\}$$

- Translation only

$${}^{A}P = {}^{B}P + {}^{A}P_{B\,org}$$

- Rotation only

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P$$

Mapping -2

- General

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{B\,org}$$

$$\underbrace{\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}_{4\times 1}}_{{}^{A}P'} = \underbrace{\left[\begin{array}{ccc:c} & {}^{A}_{B}R_{3\times 3} & & {}^{A}P_{B\,org\,3\times 1} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]}_{{}^{A}_{B}T} \underbrace{\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}_{4\times 1}}_{{}^{B}P'}$$

${}^{A}_{B}T$ homogeneous transformation

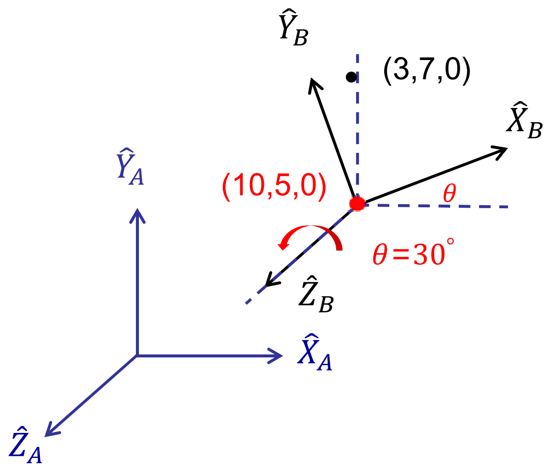
Advantage:
"sequential transformation"
Ex: ${}^{A}_{B}T = {}^{A}_{C}T\,{}^{C}_{D}T\,{}^{D}_{B}T$

$${}^{A}P' = \left[\begin{array}{ccc:c} | & | & | & | \\ {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} & {}^{A}P_{B\,org} \\ | & | & | & | \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] {}^{B}P'$$

Example

$$ {}^{B}P = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} \quad {}^{A}P_{Borg} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} \quad {}^{A}\hat{X}_B = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Y}_B = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ \begin{bmatrix} | \\ {}^{A}P \\ | \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 & 10 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} $$



看整個操作:

轉換point在不同

frame下的表達

$$ {}^{A}P = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \end{bmatrix} $$

單純看${}^{A}_{B}T$:

表達$\{B\}$相對

於{A} 的方法

Operators -1

- 前述mapping的教學運算也可做為在

  同一frame下移動和轉動的操作

- Translational operator

  - Point往前移 = frame往後移

$$ {}^{A}P_2 = {}^{A}P_1 + {}^{A}Q = D(Q)\,{}^{A}P_1 = \left[\begin{array}{ccc|c} & I & & {}^{A}Q \\ \hline 0 & 0 & 0 & 1 \end{array}\right] {}^{A}P_1 $$

- Rotational operator

  - Point逆時針轉 = frame順時針轉

$$ {}^{A}P_2 = R_{\hat{K}}(\theta)\,{}^{A}P_1 $$

  - Ex. 對$\hat{Z}$轉30° $R_{\hat{Z}}(30^\circ) = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operators -2

- Transformation operator

in {A}

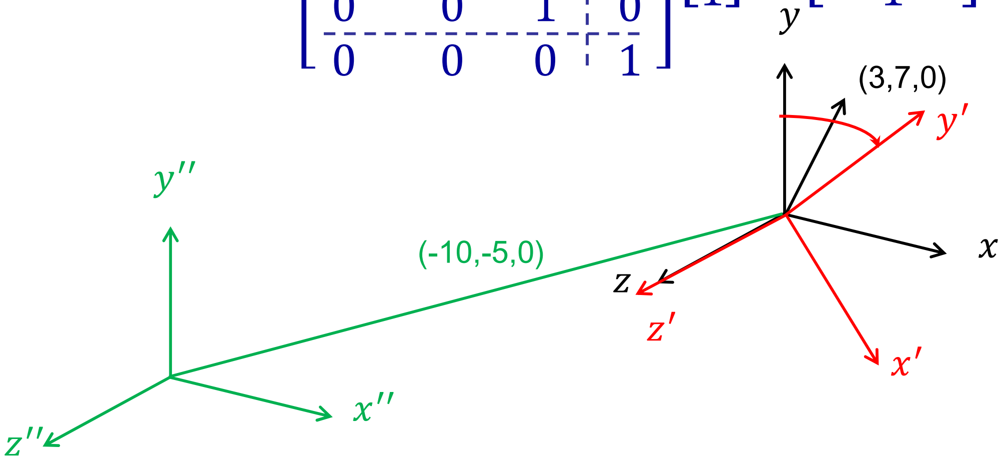
$$^{A}P_2{}' = \begin{bmatrix} ^{A}P_2 \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & R_{\hat{K}}(\theta) & & ^{A}Q \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} ^{A}P_1 \\ 1 \end{bmatrix}$$
$$= T\,^{A}P_1{}'$$

Example

point $P_1 = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$ CCW轉30°, 移動$\begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$

$$P_2{}' = TP_1{}' = \left[\begin{array}{ccc|c} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 10 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}$$

Frame的操作方向，和point的操作方向相反



Interpretations小結

- Transformation matrix的三種用法
  - 描述一個frame(相對於另一個frame)的狀態

$$
{}^{A}_{B}T = \left[\begin{array}{ccc|c} | & | & | & | \\ {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} & {}^{A}P_{B\,org} \\ | & | & | & | \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

  - 將point由某一個frame中表達mapping到另一個frame中表達

$$
{}^{A}P = {}^{A}_{B}T\,{}^{B}P
$$

  - 將point(vector)在同一個frame中進行operation（ex: 移動 & 轉動）

in $\{A\}$

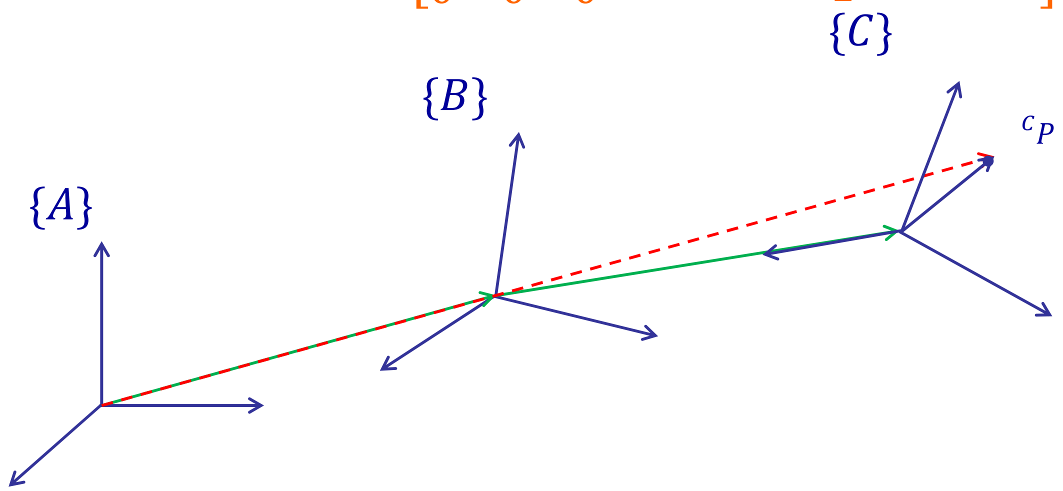
$$
{}^{A}P_{2} = T\,{}^{A}P_{1}
$$

Transformation Arithmetic -1

- Compound transformation

$$
{}^{A}P = {}^{A}_{B}T\,{}^{B}P = {}^{A}_{B}T({}^{B}_{C}T\,{}^{C}P) = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P = {}^{A}_{C}T\,{}^{C}P
$$

$$
{}^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T = \left[\begin{array}{c|c} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}_{B}R\,{}^{B}P_{Corg} + {}^{A}P_{Borg} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right]
$$



Transformation Arithmetic -2

- Inverting a transformation

$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{A}_{B}T^{-1} = {}^{B}_{A}T = \left[\begin{array}{c|c} {}^{B}_{A}R & {}^{B}P_{Aorg} \\ \hline 0 & 1 \end{array}\right] $$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T} $$

$$ 0 = {}^{B}\left({}^{A}P_{Borg}\right) = {}^{B}_{A}R\,{}^{A}P_{Borg} + {}^{B}P_{Aorg} $$

$$ \Rightarrow {}^{B}P_{Aorg} = -{}^{B}_{A}R\,{}^{A}P_{Borg} = -{}^{A}_{B}R^{T}\,{}^{A}P_{Borg} $$

$$ {}^{A}_{B}T^{-1} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Transformation Arithmetic -3

- General

$$ {}^{U}_{D}T = {}^{U}_{A}T\,{}^{A}_{D}T $$

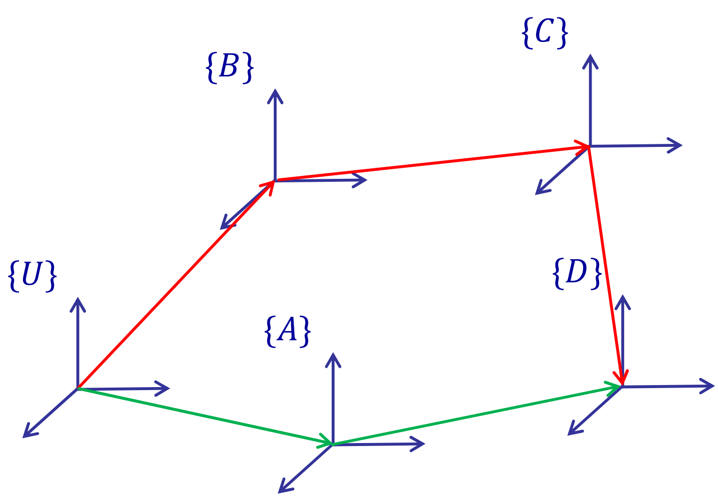
$$ {}^{U}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T $$

if ${}^{C}_{D}T$ unknown

$$ = {}^{B}_{C}T^{-1}\,{}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T $$

if ${}^{B}_{C}T$ unknown

$$ = {}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T\,{}^{C}_{D}T^{-1} $$



Transformation Arithmetic -4

- Compound transformation
  - Initial condition: $\{A\}$ and $\{B\}$ coincide $\quad {}^{A}_{B}T = I_{4\times4}$
  - $\{B\}$ rotates about the principal axes of $\{A\}$ -> Use "premultiply"
    - $\{B\}$: Rotating frame; $\{A\}$: Reference frame

以operator來想，對某一個向量，「以同一個座標基準」進行轉動或移動的操作

Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformation

$${}^{A}_{B}T = T_3 T_2 T_1 I \qquad v' = {}^{A}_{B}T v = T_3 T_2 T_1 v$$

  - $\{B\}$ rotates about the principal axes of $\{B\}$ -> Use "postmultiply"

以mapping來想，對某一個向量，從最後一個frame「逐漸轉動或移動」回第一個frame

Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformation

$${}^{A}_{B}T = I T_1 T_2 T_3$$

More on Representation of Orientation

- Cayley's formula for orthonormal matrices

$$R = (I_3 - S)^{-1}\,(I_3 + S)$$

S: skew symmetric matrix $\; -S = S^T$

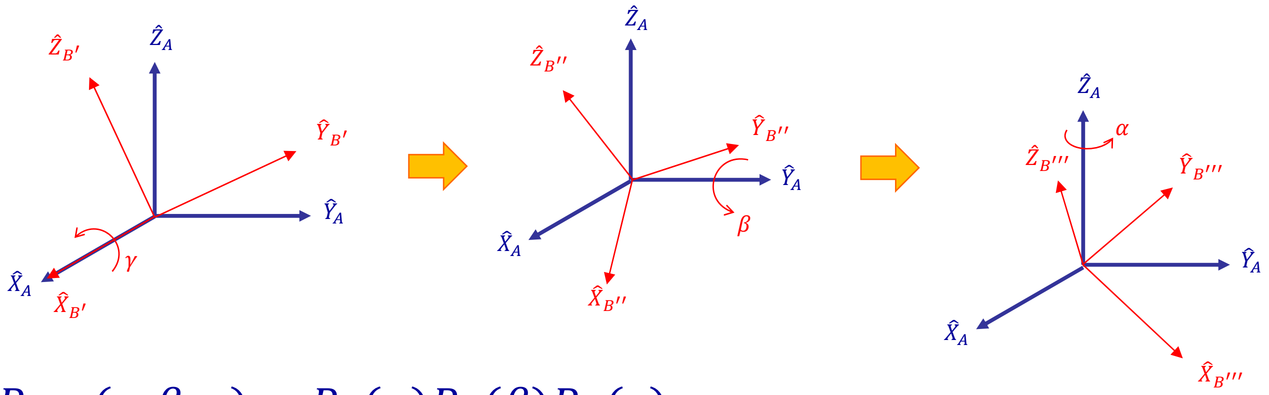
$$S = \begin{bmatrix} 0 & -S_z & S_y \\ S_z & 0 & -S_x \\ -S_y & S_x & 0 \end{bmatrix}$$

一種可以用3個參數，不經三角函數運算，即可產生rotation matrix的方法

在學完後續fixed/Euler angles後，可以想一下，這三個參數，物理上各代表什麼？

- Rotation是3 DOFs，NOT commutable
- 一般rotation matrix所表達的rotation(orientation)，標準拆解成3次旋轉的方法為何？
  - Fixed angles
  - Euler angles

X-Y-Z Fixed Angles -1



$${}^A_BR_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

X-Y-Z Fixed Angles -2

$${}^A_BR_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

If $\beta \neq 90°$

$$\beta = Atan2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$
$$\alpha = Atan2(r_{21}/c\beta, r_{11}/c\beta)$$
$$\gamma = Atan2(r_{32}/c\beta, r_{33}/c\beta)$$

$-90° \leq \beta \leq 90°$
Single solution

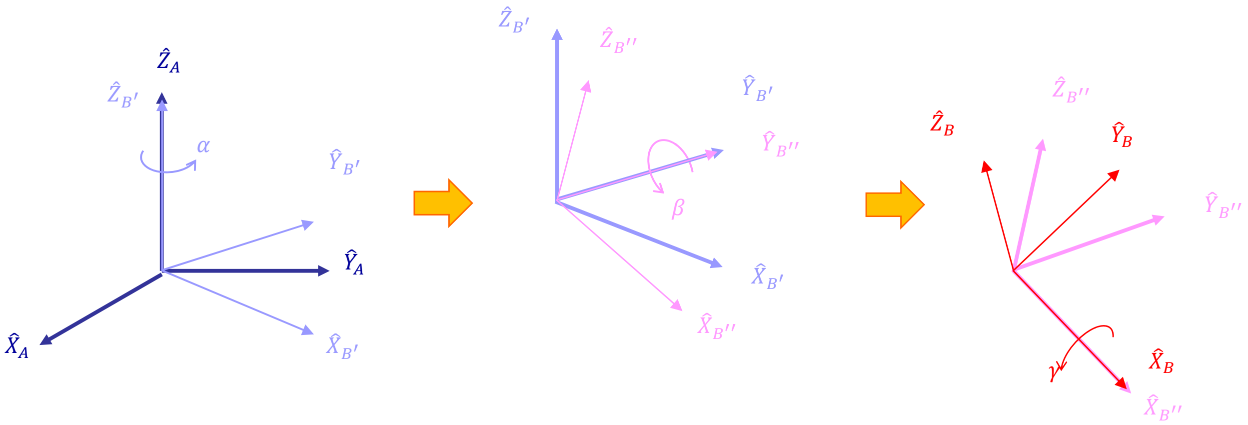
If $\beta = 90°$

$$\alpha = 0°$$
$$\gamma = Atan2(r_{12}, r_{22})$$

If $\beta = -90°$

$$\alpha = 0°$$
$$\gamma = -Atan2(r_{12}, r_{22})$$

Z-Y-X Euler Angles -1



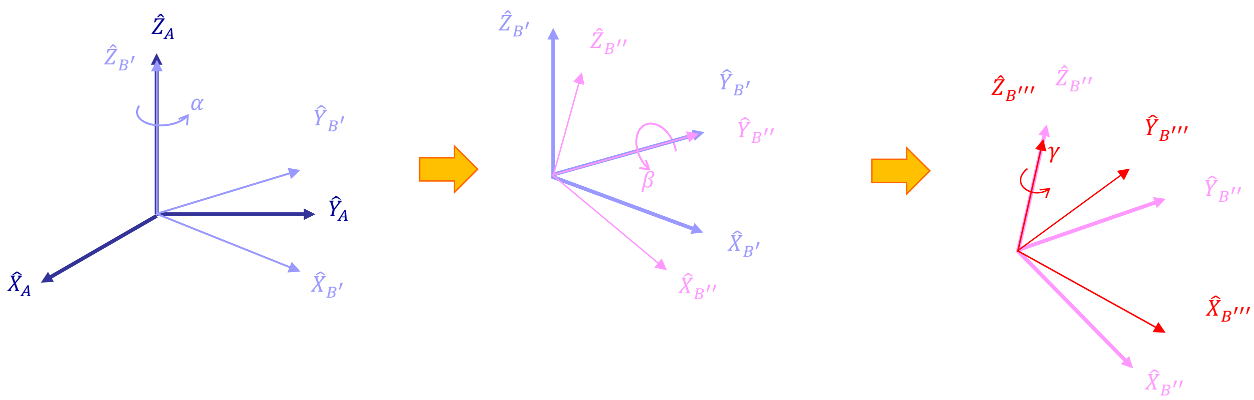
$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = {}^{A}_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) $$

"Inverse" is identical to that of the X-Y-Z Fixed angle

Z-Y-Z Euler angles -1



$$ {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) $$

$$ = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} $$

Z-Y-Z Euler angles

$$ {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} $$

If $\beta \neq 0°$

$$\beta = Atan2(\sqrt{{r_{31}}^2 + {r_{32}}^2}, r_{33})$$
$$\alpha = Atan2(r_{23}/s\beta, r_{13}/s\beta)$$
$$\gamma = Atan2(r_{32}/s\beta, -r_{31}/s\beta)$$

If $\beta = 0°$

$$\alpha = 0°$$
$$\gamma = Atan2(-r_{12}, r_{11})$$

If $\beta = 180°$

$$\alpha = 0°$$
$$\gamma = Atan2(r_{12}, -r_{11})$$

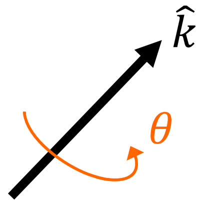
Angle-set Conventions

- 12 Fixed angle sets
- 12 Euler angle sets
- Duality: total 12 unique parametrizations of a rotation matrix by using successive rotations about principal axes

Equivalent Angle-axis Representation -1

對$\hat{k}$旋轉θ

↖ unit vector



- Rodrigues' rotation formula

$$\vec{v} \in \mathbb{R}^3$$

$$\overrightarrow{v_{rot}} = \vec{v}\cos\theta + (\hat{k} \times \vec{v})sin\theta + \hat{k}(\hat{k} \cdot \vec{v})(1 - cos\theta)$$

- Further, representing $\hat{k}$ as $K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} = \hat{k} \times$

… after derivation

"cross product matrix"

➡ $R = I + (sin\theta)K + (1 - cos\theta)K^2$

Equivalent Angle-axis Representation -2

- Thus

$$R_{\hat{k}}(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

↖ $v\theta = 1 - cos\theta$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

then $\theta = cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$

$$\hat{k} = \frac{1}{2sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$0^\circ < \theta < 180^\circ$

雙解

$R_{\hat{k}}(\theta)$ & $R_{-\hat{k}}(-\theta)$

Euler Parameters / Quaternion

- Similar to angle axis representation

define $\epsilon_1 = k_x sin\frac{\theta}{2}$ $\epsilon_2 = k_y sin\frac{\theta}{2}$ $\epsilon_3 = k_z sin\frac{\theta}{2}$

$\epsilon_4 = cos\frac{\theta}{2}$

note $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$ 4個參數+1個限制條件，所以也是3 DOFs

quaternion

$$q = \epsilon_4 + \epsilon_1\hat{\imath} + \epsilon_2\hat{\jmath} + \epsilon_3\hat{k}$$

$$= cos\frac{\theta}{2} + sin\frac{\theta}{2}(k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k})$$

以quaternion方式表達的旋轉有特殊的操作方式，相較於Rotation Matrix有效率很多，有興趣的同學可以上網找相關說明。

Euler Parameters / Quaternion

$$R_\epsilon(\theta) = \begin{bmatrix} 1-2\epsilon_2^2-2\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3+\epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2+\epsilon_3\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3-\epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3+\epsilon_1\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_2^2 \end{bmatrix}$$

"inverse"

$$\epsilon_1 = \frac{r_{32} - r_{23}}{4\epsilon_4}$$

$$\epsilon_2 = \frac{r_{13} - r_{31}}{4\epsilon_4}$$

$$\epsilon_3 = \frac{r_{21} - r_{12}}{4\epsilon_4}$$

$$\epsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

終

- Questions?