Chap 5 The Performance of Feedback Control Systems

林沛群
國立台灣大學
機械工程學系

Introduction

- A feedback control system can
  - Adjust the transient and steady-state responses
  - Reduce system sensitivities to the parameter variation
  - Disturbance / noise rejection

- Design specifications (for control system)

  For a specified input command
  - Several time response indices
  - Steady-state accuracy

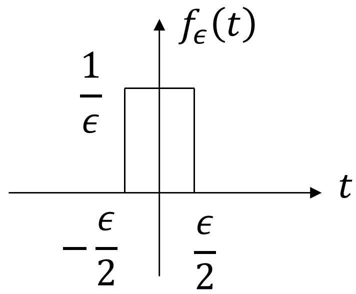
Test Input Signals

- Why do we want to do this ?
  - Actual input signal is usually unknown
  - Provide several standard measures of the performance of the system
  - Exist a reasonable correlation between the response of a system to a standard test input and an actual input

Some standard test signals -1

- Unit impulse $R(s) = 1$
  - Rectangular function

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & -\frac{\epsilon}{2} \le t \le \frac{\epsilon}{2} \\ 0 & otherwise \end{cases}$$



  - Unit impulse

$$\delta(t) \triangleq \lim_{\epsilon \to 0} f_\epsilon(t)$$

    - Properties

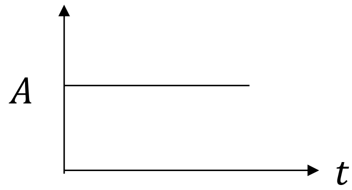
$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \int_{-\infty}^{\infty} \delta(t-a)g(t)dt = g(a)$$

$$y(t) = \mathcal{L}^{-1}[G(s)R(s)] = \mathcal{L}^{-1}[G(s)] = g(t)$$

Some standard test signals -2

- Step $R(s) = \dfrac{A}{s}$

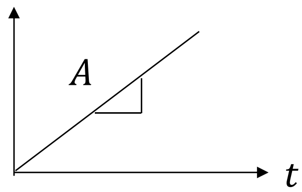
$$r(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$y(\infty) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s)\frac{1}{s} = G(0) \quad (if\ A = 1)$$

- Ramp $R(s) = \dfrac{A}{s^2}$

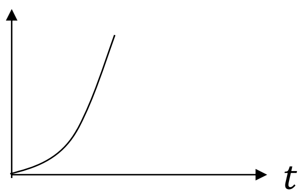
$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Some standard test signals -3

- Parabolic $R(s) = \dfrac{2A}{s^3}$

$$r(t) = \begin{cases} At^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



- General form (polynomial) $R(s) = \dfrac{n!}{s^{n+1}}$

$$r(t) = \begin{cases} t^n & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Example

- A 1st-order system $G(s) = \dfrac{9}{s+10}$

  Unit step input $r(t) = 1$

$$Y(s) = G(s)R(s) = \frac{9}{s(s+10)} = \frac{0.9}{s} + \frac{-0.9}{s+10}$$

$$y(t) = 0.9(1 - e^{-10t}) \Rightarrow y(\infty) = 0.9$$

$$e_{ss} = e(\infty) = 0.1$$

Or $\Rightarrow$

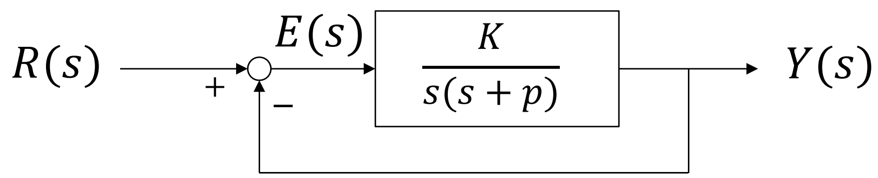
$$y(\infty) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0) = 0.9$$

$$e_{ss} = e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\big(1 - G(s)\big)\frac{1}{s}$$

$$= 1 - 0.9 = 0.1$$

Performance of Second-order Systems -1

- Consider a single-loop second-order system



$$Y(s) = \frac{G(s)}{1+G(s)}R(s) = \frac{K}{s^2 + ps + K}R(s)$$

$$= \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}R(s)$$

DC gain = 1

ζ: damping ratio, "zeta"

ω_n: natural frequency

Performance of Second-order Systems -2

- Assuming unit step input $R(s) = \frac{1}{s}$

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}\frac{1}{s}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \left(\frac{z_1}{s - p_1} + \frac{z_2}{s - p_2}\right)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = 1 - z_1 e^{p_1 t} - z_2 e^{p_2 t}$$

Performance of Second-order Systems -3

- When $\zeta = 0$

$$\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s}{s^2 + \omega_n^2} = \frac{z_1}{s - p_1} + \frac{z_2}{s - p_2}$$

$$p_{1,2} = \pm j\omega_n \quad \Rightarrow \quad z_1 = \left.\frac{s}{s - p_2}\right|_{s = p_1} = \frac{1}{2}, \quad z_2 = \left.\frac{s}{s - p_1}\right|_{s = p_2} = \frac{1}{2}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\Rightarrow y(t) = 1 - z_1 e^{p_1 t} - z_2 e^{p_2 t} = 1 - \frac{1}{2}e^{j\omega_n t} - \frac{1}{2}e^{-j\omega_n t}$$

$$= 1 - \cos(\omega_n t)$$

Performance of Second-order Systems -4

- When $0 < \zeta < 1$

$$\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{z_1}{s-p_1} + \frac{z_2}{s-p_2}$$

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = (-\zeta \pm j\beta)\omega_n$$

where $\beta \equiv \sqrt{1-\zeta^2}$

$$z_1 = \left.\frac{s+2\zeta\omega_n}{s-p_2}\right|_{s=p_1} = \frac{\beta - j\zeta}{2\beta}$$

$$z_2 = \left.\frac{s+2\zeta\omega_n}{s-p_1}\right|_{s=p_2} = \frac{\beta + j\zeta}{2\beta}$$

Performance of Second-order Systems -5

- $\downarrow \mathcal{L}^{-1}$

$$y(t) = 1 - z_1 e^{p_1 t} - z_2 e^{p_2 t}$$

Euler's formula $e^{j\theta} = cos\theta + jsin\theta$

$$= 1 - z_1 e^{-\zeta\omega_n t}(\cos(\omega_n\beta t) + jsin(\omega_n\beta t))$$
$$-z_2 e^{-\zeta\omega_n t}(\cos(\omega_n\beta t) - jsin(\omega_n\beta t))$$

$$= 1 - \frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n\beta t + \theta)$$

$\beta = \sqrt{1-\zeta^2} \triangleq sin\theta$

$\therefore \theta = \cos^{-1}\zeta$

Performance of Second-order Systems -6

- When $\zeta = 1$

$$\frac{s+2\omega_n}{s^2+2\omega_n s+\omega_n^2} = \frac{1}{s+\omega_n} + \frac{\omega_n}{(s+\omega_n)^2}$$

$$p_{1,2} = -\omega_n \quad double\ pole$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} = 1 - (1+\omega_n t)e^{-\omega_n t}$$

Note: $\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$

Repeated root introduce slower response

Performance of Second-order Systems -7

- When $\zeta > 1$

$$\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{z_1}{s-p_1} + \frac{z_2}{s-p_2}$$

$$p_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2-1}\right)\omega_n$$

$$z_1 = \left.\frac{s+2\zeta\omega_n}{s-p_2}\right|_{s=p_1} = \frac{\sqrt{\zeta^2-1}+\zeta}{2\sqrt{\zeta^2-1}} > 0$$

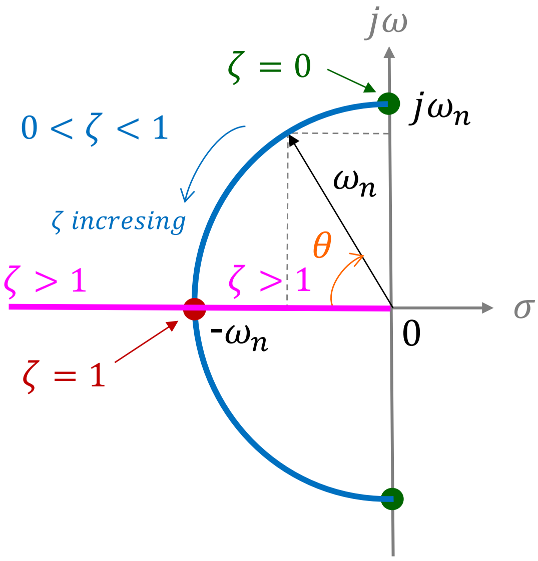
$$z_2 = \left.\frac{s+2\zeta\omega_n}{s-p_1}\right|_{s=p_2} = \frac{\sqrt{\zeta^2-1}-\zeta}{2\sqrt{\zeta^2-1}} < 0$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = 1 - z_1 e^{p_1 t} - z_2 e^{p_2 t}$$ 等同兩個一階系統反應的線性合成

Performance of Second-order Systems -8

- The locus of roots as ζ varies with ω_n constant.



$\theta = \cos^{-1}\zeta$

$\zeta = 0$

$$p_{1,2} = \pm j\omega_n$$

$0 < \zeta < 1$

$$p_{1,2} = \left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_n$$

$\zeta = 1$

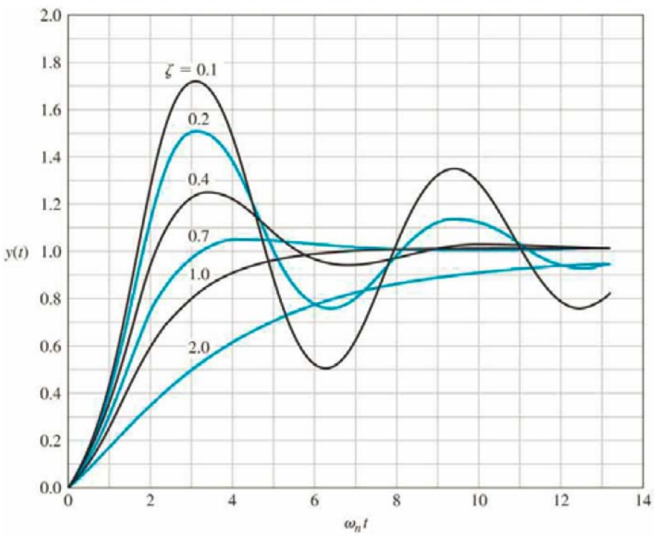
$$p_{1,2} = -\omega_n$$

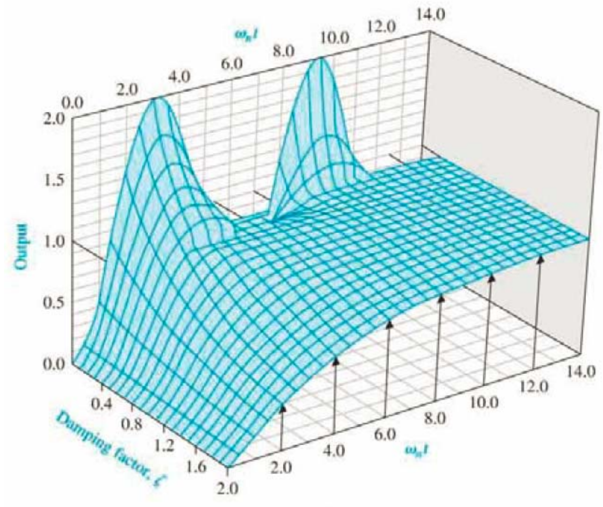
$\zeta > 1$

$$p_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2-1}\right)\omega_n$$

Performance of Second-order Systems -9

- Step response as ζ varies





$\zeta = 0$ $\quad y(t) = 1 - \cos(\omega_n t)$

$0 < \zeta < 1$ $\quad y(t) = 1 - \frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n \beta t + \theta)$

$\zeta = 1$ $\quad y(t) = 1 - (1 + \omega_n t)e^{-\omega_n t}$

Performance of Second-order Systems -10

- Assuming impulse input $R(s) = 1$

$$Y_2(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{z_1}{s - p_1} + \frac{z_2}{s - p_2}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = g(t) = z_1 e^{p_1 t} + z_2 e^{p_2 t}$$ 可用前幾頁描述之方式求解

或者，以前幾頁所得之解為基礎

$$Y_2(s) = sY(s) \quad \Rightarrow \quad y_2(t) = \frac{d}{dt} y(t)$$

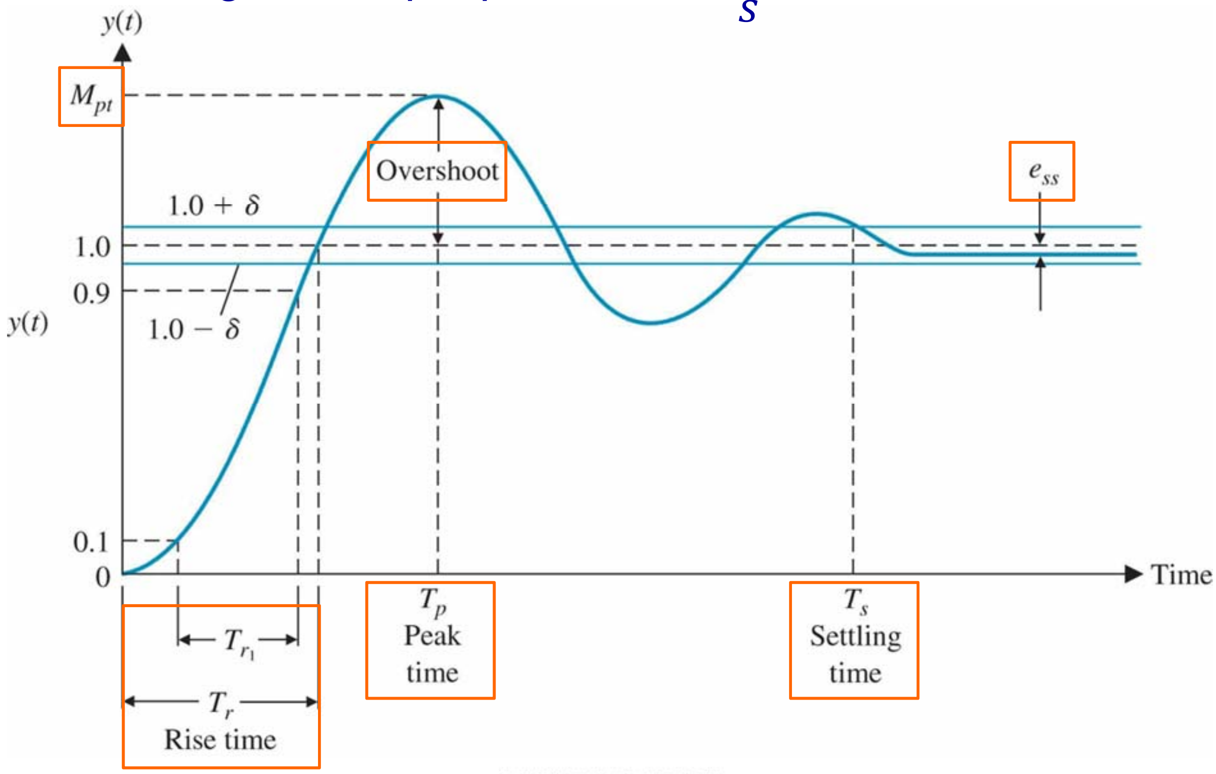
以$0 < \zeta < 1$為例

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

$$y_2(t) = \frac{1}{\beta} \omega_n e^{-\zeta\omega_n t} \sin(\omega_n \beta t)$$

Response of Second-order Systems -1

- Assuming unit step input $R(s) = \frac{1}{s}$



Response of Second-order Systems -2

- Settling time T_s
  - The time required for the system to settle within a certain percentage δ of the input amplitude
  - For standard 2nd-order systems

If $0 < \zeta < 1$ and $\delta = 2\%$

$$y(t) = 1 - e^{-\zeta\omega_n t}[\frac{1}{\beta}\sin(\omega_n\beta t + \theta)]$$

$$e^{-\zeta\omega_n T_s} < 0.02$$

$$\zeta\omega_n T_s \sim 4$$

$$\Rightarrow \quad T_s = \frac{4}{\zeta\omega_n} = 4\tau$$

Response of Second-order Systems -3

- Peak time T_p
  - Time at which the amplitude is largest
- Percentage overshoot $P.O.$
  - $$P.O. = \frac{M_{pt} - f_v}{f_v} \times 100\%$$

M_{pt}: peak value of the time response

f_v: final value of the response
(may not be the same as the input)

Response of Second-order Systems -4

- For standard 2nd-order systems

If $0 < \zeta < 1$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

$$\dot{y(t)} = \frac{1}{\beta} \omega_n e^{-\zeta\omega_n t} \sin(\omega_n \beta t) = 0$$ (i.e. impulse response)

$$\omega_n \beta t = n\pi$$ When $n = 1 \rightarrow t = T_p$

$$\omega_n \beta T_p = \pi$$

$$\Rightarrow T_p = \frac{\pi}{\omega_n \beta} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

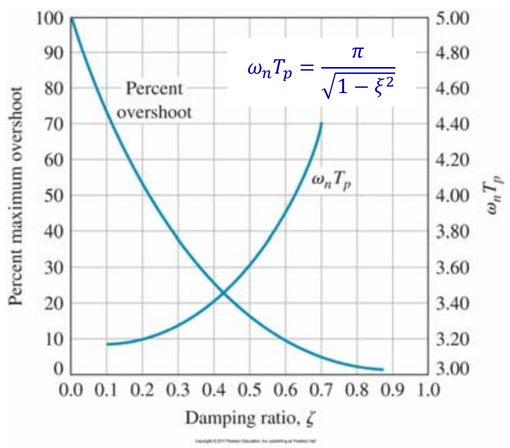
Response of Second-order Systems -5

$$y(T_p) = M_{pt} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$ Independent of ω_n

$$f_v = 1$$

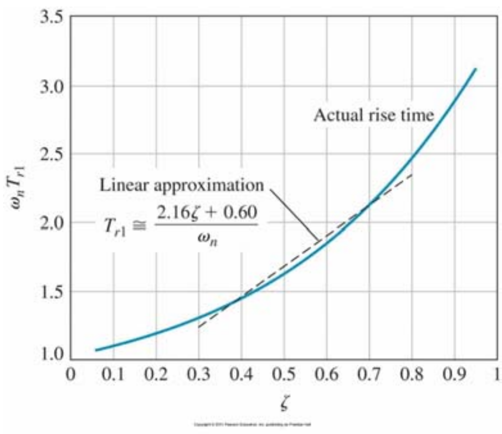
$$\Rightarrow P.O. = \frac{M_{pt} - f_v}{f_v} \times 100\% = 100 e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\Rightarrow \text{OR} \quad \xi = \sqrt{\frac{\left[l_n\left(\frac{P.O.}{100}\right)\right]^2}{\pi^2 + \left[l_n\left(\frac{P.O.}{100}\right)\right]^2}}$$



Response of Second-order Systems -6

- Rise time T_r
  - For underdamped systems: $T_r\ 0-100\%$
  - For overdamped/underdamped systems: $T_{r_1}\ 10-90\%$



Linear approximation

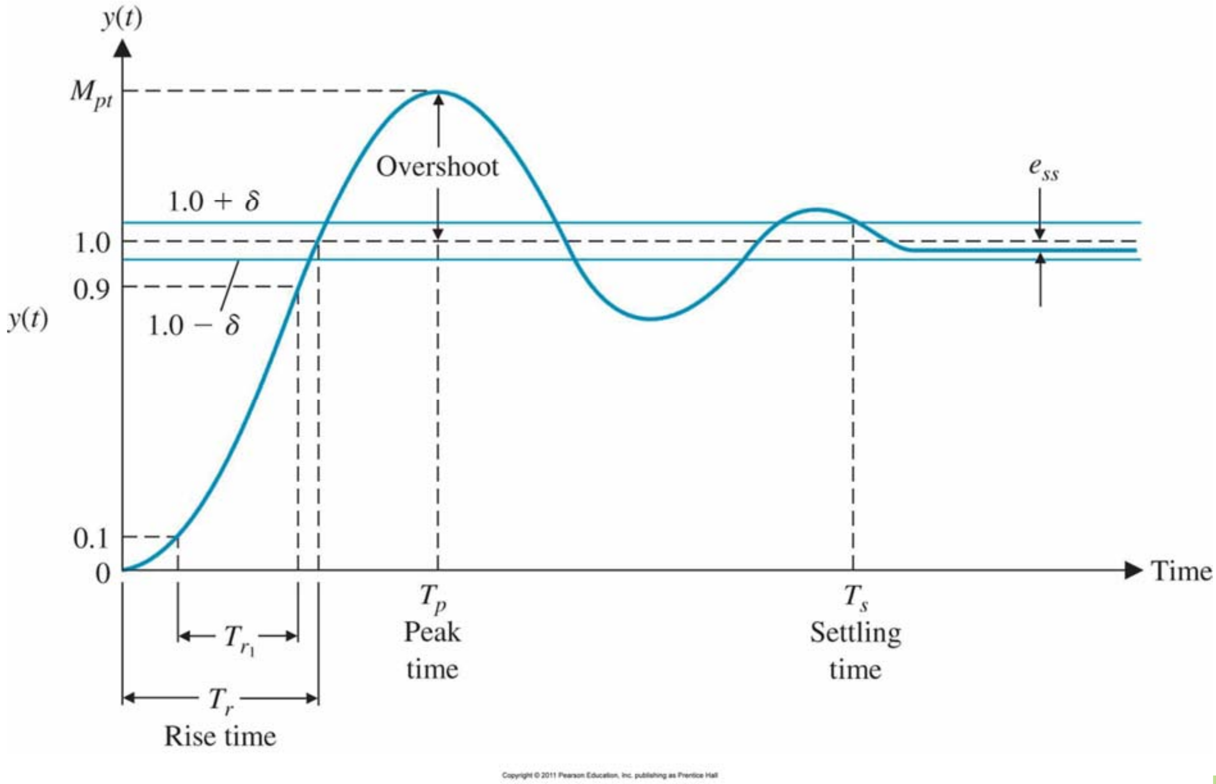
$$T_{r_1} \sim \frac{(2.16\zeta + 0.6)}{\omega_n}$$

$$for\ 0.3 \le \zeta \le 0.8$$

$larger\ \omega_n$ and $smaller\ \zeta$ ⇨ $faster\ response$, $larger\ overshoot$

Response of Second-order Systems -7

- Versus time



Response of Second-order Systems -8

- Comparison

	Dorf	Kuo	Franklin
T_s (Want $\zeta\omega_n \uparrow$)	$\sim \frac{4}{\zeta\omega_n}$ (2%)	$\sim \frac{3.2}{\zeta\omega_n} (0<\zeta<0.69)$ $\sim \frac{4.53}{\zeta\omega_n} (\zeta>0.69)$ (2%)	$\sim \frac{4.6}{\zeta\omega_n}$ (1%)
T_{r_1} (Want $\omega_n \uparrow$)	$\sim \frac{(2.16\zeta+0.6)}{\omega_n}$ $for\ 0.3 \le \zeta \le 0.8$	$\sim \frac{(2.5\zeta+0.8)}{\omega_n}$ $\sim \frac{(1-0.4167\zeta+2.917\zeta^2)}{\omega_n}$ $for\ 0<\zeta<1$	$\sim \frac{1.8}{\omega_n}$ $for\ \zeta \sim 0.5$
T_p		$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$	
$P.O.$ (Want $\zeta \uparrow$)		$100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$	

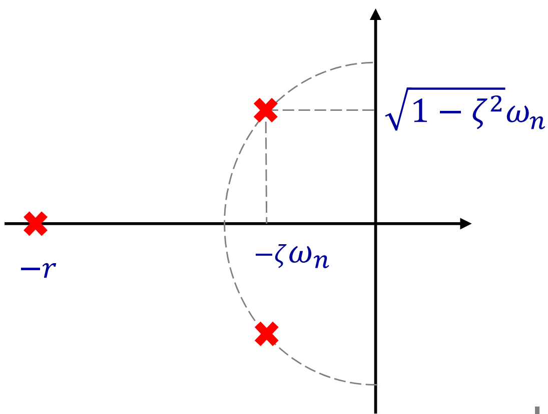
Effect of a Third Pole and a Zero -1

- Add a pole on the 2nd-order system

$$T(s) = \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}\frac{r}{(s+r)}$$

$\downarrow \mathcal{L}^{-1}$ poles $p_{1,2,3} = \left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_n, -r$

$$t(t) = z_1 e^{p_1 t} + z_2 e^{p_2 t} + z_3 e^{p_3 t}$$



If
$|r| \ge 10|\zeta\omega_n|$,
the system can be approximated by the 2nd-order system because the effect of the 3rd root decays vary fast comparing to the other two roots

Effect of a Third Pole and a Zero -2

- Add a zero on the 2nd-order system

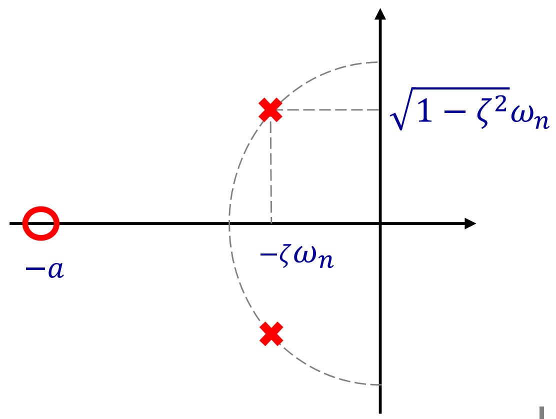
$$T(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \frac{(s+a)}{a}$$

$\downarrow \mathcal{L}^{-1}$

$$t(t) = z_1 e^{p_1 t} + z_2 e^{p_2 t}$$

poles $p_{1,2} = (-\zeta \pm j\sqrt{1-\zeta^2})\omega_n$

zero $z = -a$



- Change weights of all poles
- May cause pole-zero cancellation

Effect of a Third Pole and a Zero -3

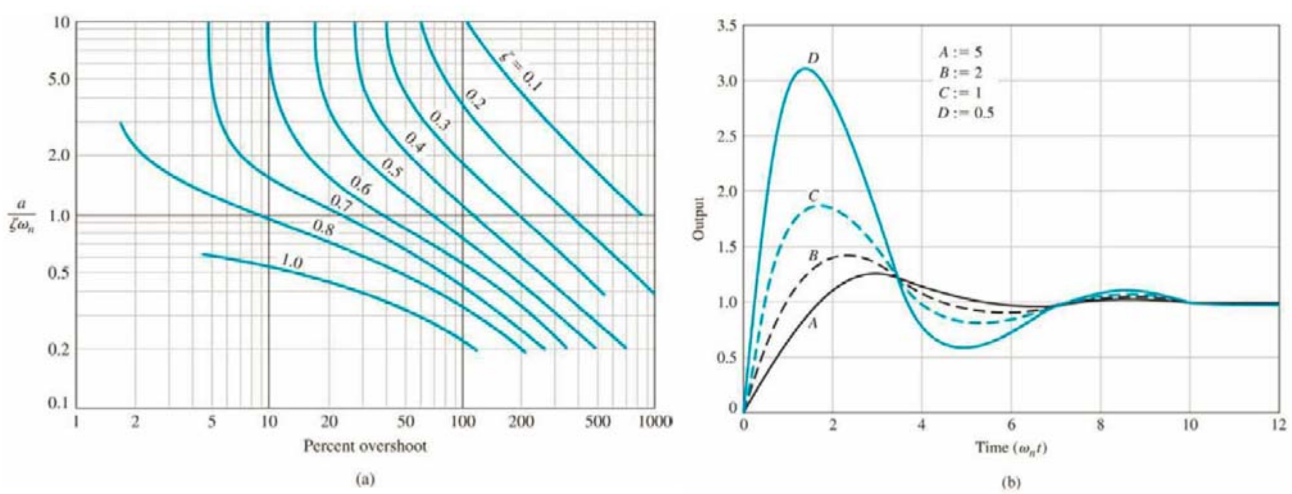


Figure 5.13 (a) Percent overshoot as a function of ζ and ω_n when a second-order transfer function contains a zero. Redrawn with permission from R. N. Clark, *Introduction to Automatic Control* Systems (New York: Wiley, 1962). (b) The response for the second-order transfer function with a zero for four values of the ratio $a/\zeta\omega_n$: $A = 5$, $B = 2$, $C = 1$, and $D = 0.5$ when $\zeta = 0.45$.

Effect of a Third Pole and a Zero -4

- Assuming $a = \alpha\zeta\omega_n$

$$T(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \frac{(s + \alpha\zeta\omega_n)}{\alpha\zeta\omega_n}$$

$$= \frac{\frac{1}{\alpha\zeta}\left(\frac{s}{\omega_n}\right)+1}{\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1} \qquad \text{Set } \bar{s} = \frac{s}{\omega_n}$$

$$= \frac{\frac{\bar{s}}{\alpha\zeta}+1}{\bar{s}^2 + 2\zeta\bar{s} + 1} = \frac{1}{\bar{s}^2 + 2\zeta\bar{s} + 1} + \frac{1}{\alpha\zeta}\bar{s}\frac{1}{\bar{s}^2 + 2\zeta\bar{s} + 1}$$

Original 2nd-order system

Derivative of the original 2nd-order system

If α↑, effect↓

In general, *LHP zeros increase P.O.* and *RHP zeros decrease P.O.*.

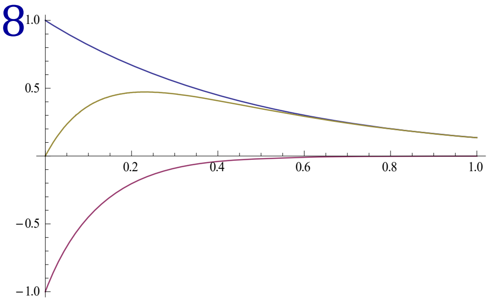
Effect of a Third Pole and a Zero -5

- Example:

$$G_1(s) = \frac{6}{s^2 + 10s + 16} = \frac{1}{s+2} + \frac{-1}{s+8}$$

impulse response

$$y_1(t) = g_1(t) = e^{-2t} - e^{-8t}$$

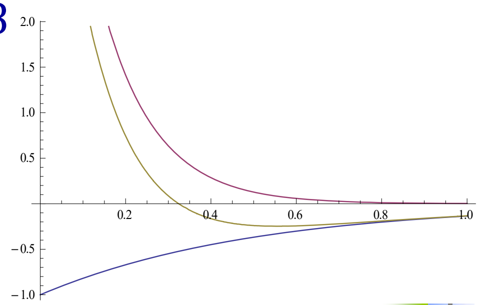


Add a zero

$$G_2(s) = \frac{6s + 6}{s^2 + 10s + 16} = \frac{-1}{s+2} + \frac{7}{s+8}$$

impulse response

$$y_2(t) = g_2(t) = -e^{-2t} + 7e^{-8t}$$
$$= y_1(t) + \dot{y}_1(t)$$



The s-plane Root Locations -1

- General T.F.

$$T(s)=\frac{Y(s)}{R(s)}=\frac{\sum P_i\Delta_i(s)}{\Delta(s)}=\frac{p(s)}{q(s)}$$

Roots of $p(s)=0$: zeros

Roots of $q(s)=0$: poles

In general, if $R(s)=\frac{1}{s}$, $T(0)=1$, and no repeated roots

$$Y(s)=\frac{1}{s}+\sum_{i=1}^{M}\frac{A_i}{s+\sigma_i}+\sum_{k=1}^{N}\frac{B_k s+C_k}{s^2+2\alpha_k s+(\alpha_k^2+\omega_k^2)}$$

$$A_i, B_k, C_k = constants$$

$$s=-\sigma_i, -\alpha_k \pm j\omega_k$$

The s-plane Root Locations -2

$\downarrow \mathcal{L}^{-1}$

$$y(t)=1+\sum_{i=1}^{M}A_i e^{-\sigma_i t}+\sum_{k=1}^{N}D_k e^{-\alpha_k t}\sin(\omega_k t+\theta_k)$$

$$D_k=f(B_k, C_k, \alpha_k, \omega_k)=constant$$

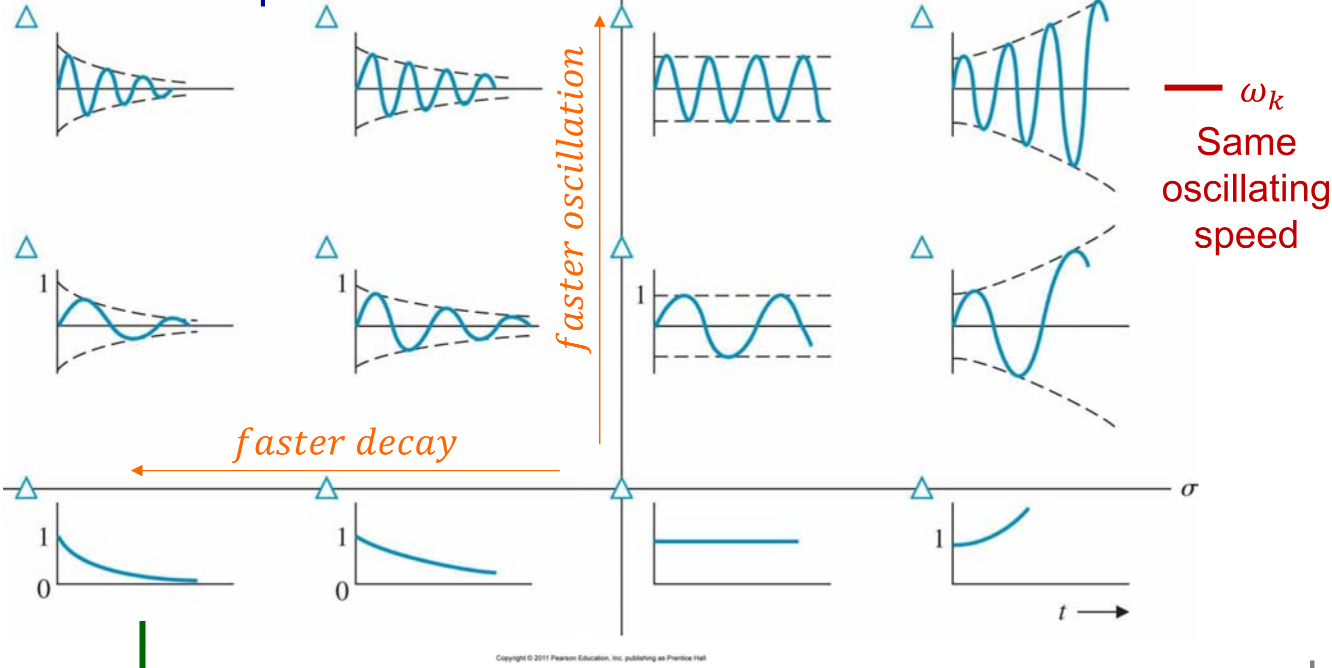
➡ Locations of poles and zeros determines the system response

For the response to be stable,

$-\sigma_i, -\alpha_k$ must be in the left-hand s-plane (LHP)

The s-plane Root Locations -3

- Impulse response for various root locations in the s-plane



$-\sigma_i\ or\ -\sigma_k$ Same decay speed

The s-plane Root Locations -4

- Comments
  - System performance – in time domain y(t)

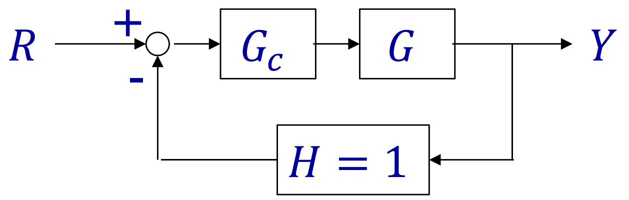
    System analysis and design – in frequency domain pole & zero
  - Poles of T(s) determine the particular response modes

    Zeros of T(s) establish the relative weightings of individual modes

    reduce the effect of nearby poles
  - The response of a complex linear system can be thought as the combination of these simple 1st-order and 2nd-order system responses

Steady-state Error of Feedback Control Systems -1

- A standard closed-loop system



Considering unity feedback

$$G_cG(s) = \frac{k\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}$$

N: type number

$$E(s) = \frac{1}{1+G_cG}R(s)$$

$$e_{ss} = \lim_{t\to\infty} E(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+G_cG}R(s)$$

Steady-state Error of Feedback Control Systems -2

- Step input $R(s) = \frac{A}{s}$

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+G_c(s)G(s)}\frac{A}{s} = \lim_{s\to 0}\frac{A}{1+G_c(s)G(s)}$$

$$= \lim_{s\to 0}\frac{As^N\prod_{k=1}^{Q}(s+p_k)}{s^N\sum_{k=1}^{Q}(s+p_k) + k\prod_{i=1}^{M}(s+z_i)}$$

$$if\ N \geq 1 \quad e_{ss} = 0$$

$$if\ N = 0 \quad e_{ss} = \frac{A}{1+k_p}$$

$$k_p \triangleq \lim_{s\to 0} G_cG(s)$$

position error constant

Steady-state Error of Feedback Control Systems -3

- Ramp input $R(s) = \frac{A}{s^2}$

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+G_c(s)G(s)} \frac{A}{s^2} = \lim_{s\to 0} \frac{A}{sG_c(s)G(s)}$$

$$= \lim_{s\to 0} \frac{As^{N-1}\prod_{k=1}^{Q}(s+p_k)}{k\prod_{i=1}^{M}(s+z_i)}$$

$if\ N \geq 2 \quad e_{ss} = 0$

$if\ N = 1 \quad e_{ss} = \frac{A}{k_v}$ $k_v \triangleq \lim_{s\to 0} sG_c(s)G(s)$

$if\ N = 0 \quad e_{ss} = \infty$ *velocity error constant*

Steady-state Error of Feedback Control Systems -4

- Parabola input $R(s) = \frac{A}{s^3}$

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+G_c(s)G(s)} \frac{A}{s^3} = \lim_{s\to 0} \frac{A}{s^2G_c(s)G(s)}$$

$$= \lim_{s\to 0} \frac{As^{N-2}\prod_{k=1}^{Q}(s+p_k)}{k\prod_{i=1}^{M}(s+z_i)}$$

$if\ N \geq 3 \quad e_{ss} = 0$

$if\ N = 2 \quad e_{ss} = \frac{A}{k_a}$ $k_a \triangleq \lim_{s\to 0} s^2G_c(s)G(s)$

$if\ N = 1 \quad e_{ss} = \infty$ *acceleration error constant*

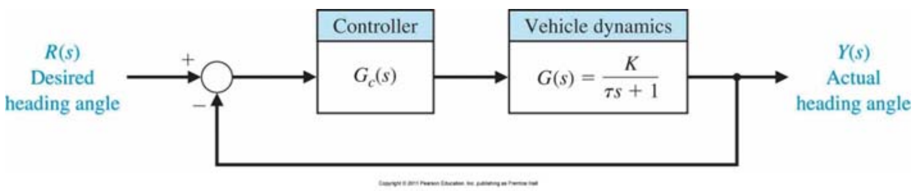
Steady-state Error of Feedback Control Systems -5

- Summary

$Type\ number\ of\ G_cG$	$Step$	$Ramp$	$Parabola$
0	$\frac{A}{1+k_p}$	∞	∞
1	0	$\frac{A}{k_v}$	∞
2	0	0	$\frac{A}{k_a}$

Steady-state Error of Feedback Control Systems -6

- Example: Mobile robot steering control



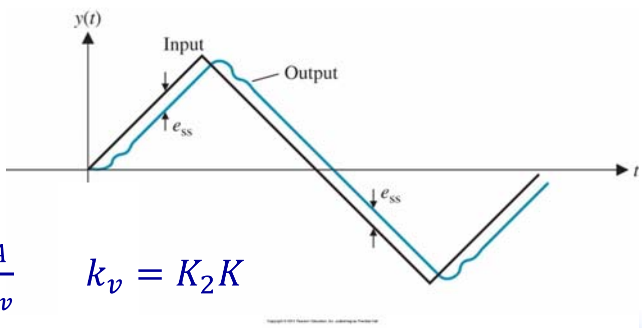
$G_c(s) = K_1$

$N = 0$ Step input: $e_{ss} = \frac{A}{1+k_p}$ $k_p = K_1 K$

$G_c(s) = K_1 + \frac{K_2}{s}$

$N = 1$ Step input: $e_{ss} = 0$

Ramp input: $e_{ss} = \frac{A}{k_v}$ $k_v = K_2 K$



Performance Indices -1

- Definition
  - A performance index is a quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system specifications

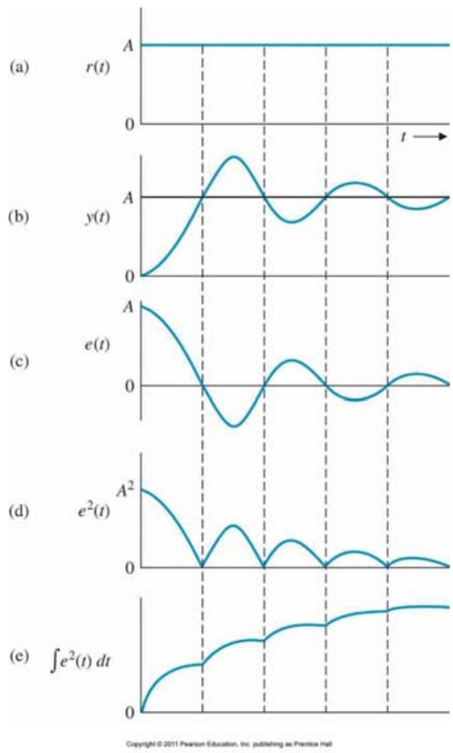
- Optimum control system
  - When the system parameters are adjusted so that the index reaches an extreme, commonly a minimum value

Performance Indices -2

- ISE, integral of the square of the error

$$T = T_s \quad (usually)$$

$$ISE = \int_0^T e^2(t)dt$$



  - Can discriminate excessively over-damped systems and excessively underdamped systems
  - Can be solved analytically

$$\frac{d\ ISE}{dK} = 0 \ \ to\ get\ K$$

Performance Indices -3

- IAE, integral of the absolute magnitude of the error

$$IAE = \int_0^T |e(t)|dt$$

- ITSE, integral of time multiplied by the squared error

$$ITSE = \int_0^T te^2(t)dt$$

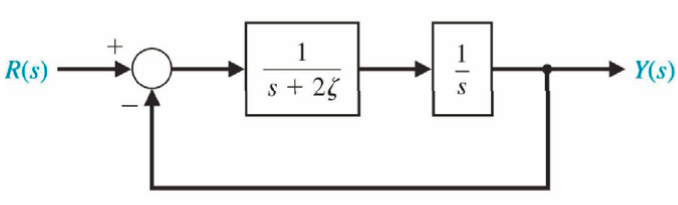
To reduce the contribution of the large initial error

- ITAE, integral of time multiplied by absolute error, ITAE

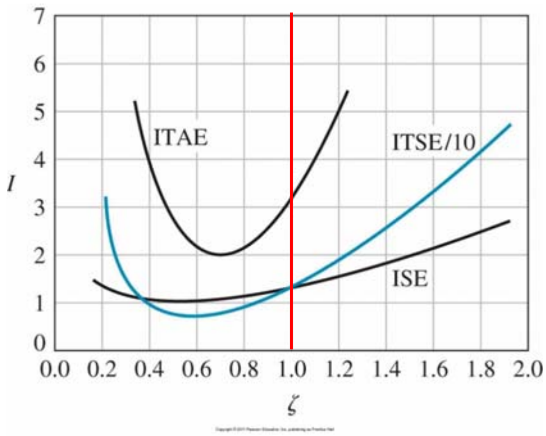
$$ITAE = \int_0^T t|e(t)|dt$$

Performance Indices -4

- Example $R(s) = \frac{1}{s}$



$$T(s) = \frac{1}{s^2 + 2\xi s + 1}$$



- The indices of the critical damped system is always smaller than those of the overdamped system
- Minimum occurs in the underdamped system---going up fast but with oscillation

Performance Indices -5

- Optimum coefficients of $T(s)$ based on ITAE criterion for a step input

$$T(s)=\frac{Y(s)}{R(s)}=\frac{b_0}{s^n+b_{n-1}s^{n-1}+\ \dots+b_1s+b_0}$$

$$=\frac{\dfrac{b_0}{s^n+b_{n-1}s^{n-1}+\ \dots+b_1s}}{1+\dfrac{b_0}{s^n+b_{n-1}s^{n-1}+\ \dots+b_1s}}$$

$N=1$

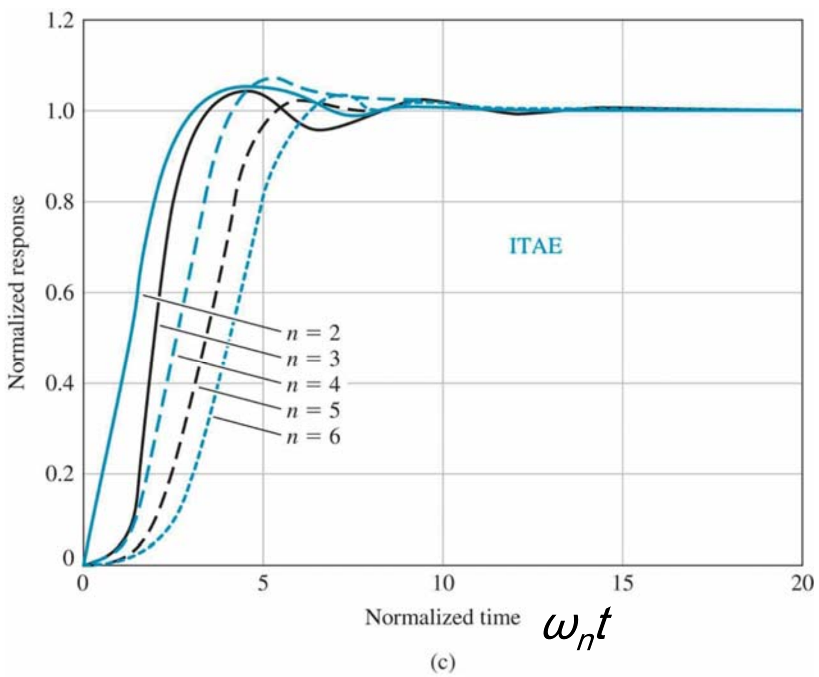
$e_{ss}=0$

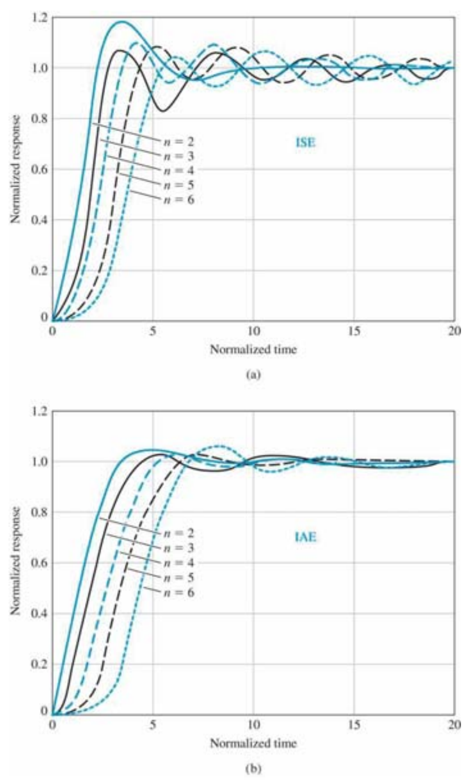
Table 5.6 The Optimum Coefficients of *T*(s) Based on the ITAE Criterion for a Step Input

$s+\omega_n$
$s^2+1.4\omega_n s+\omega_n^2$ → $\zeta=0.7$
$s^3+1.75\omega_n s^2+2.15\omega_n^2 s+\omega_n^3$
$s^4+2.1\omega_n s^3+3.4\omega_n^2 s^2+2.7\omega_n^3 s+\omega_n^4$
$s^5+2.8\omega_n s^4+5.0\omega_n^2 s^3+5.5\omega_n^3 s^2+3.4\omega_n^4 s+\omega_n^5$
$s^6+3.25\omega_n s^5+6.60\omega_n^2 s^4+8.60\omega_n^3 s^3+7.45\omega_n^4 s^2+3.95\omega_n^5 s+\omega_n^6$

Performance Indices -6

- Step responses of a normalized transfer function





Performance Indices -7

- Optimum coefficients of $T(s)$ based on ITAE criterion for a ramp input

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_1 s + b_0}{s^n + b_{n-1}s^{n-1} + \ ... + b_1 s + b_0}$$

$$= \frac{\dfrac{b_1 s + b_0}{s^n + b_{n-1}s^{n-1} + \ ... + b_2 s^2}}{1 + \dfrac{b_1 s + b_0}{s^n + b_{n-1}s^{n-1} + \ ... + b_2 s^2}}$$

$N = 2$

$e_{ss} = 0$

Table 5.7 The Optimum Coefficients of *T*(s) Based on the ITAE Criterion for a Ramp Input

$s^2 + 3.2\omega_n s + \omega_n^2$
$s^3 + 1.75\omega_n s^2 + 3.25\omega_n^2 s + \omega_n^3$
$s^4 + 2.41\omega_n s^3 + 4.93\omega_n^2 s^2 + 5.14\omega_n^3 s + \omega_n^4$
$s^5 + 2.19\omega_n s^4 + 6.50\omega_n^2 s^3 + 6.30\omega_n^3 s^2 + 5.24\omega_n^4 s + \omega_n^5$

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- Questions?